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MATTER AND MOTION.

BY

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PREFACE TO AMERICAN EDITION

THE eminent position of the author of this essay, and the importance which the questions here discussed have recently assumed, seemed to demand an increase of facilities for obtaining the work in this country.

The interest manifested in the discussions of kindred subjects in VAN NOSTRAND'S MAGAZINE, induced the publisher to lay this little treatise before the readers of that periodical. It was, therefore, published in the April and May numbers of the Magazine, and is here re-printed entire, with some changes in mechanical arrangement and execution of cuts and typography from the foreign edition, which are believed to be much to the advantage of the reader.

The importance of such clear conceptions of the fundamental principles of physics as are afforded by so eminent a writer is unquestionable.

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MATTER AND MOTION.

CHAPTER I.

INTRODUCTION.

NATURE OF PHYSICAL SCIENCE.—Physical science is that department of knowledge which relates to the order of nature, or, in other words, to the regular succession of events.

The name of physical science, however, is often applied in a more or less restricted manner to those branches of science in which the phenomena considered are of the simplest and most abstract kind, excluding the consideration of the more complex phenomena, such as those observed in living beings.

The simplest case of all is that in which an event or phenomenon can be described as a change in the arrangement of certain bodies. Thus the motion

of the moon may be described by stating the changes in her position relative to the earth in the order in which they follow one another.

In other cases we may know that some change of arrangement has taken place, but we may not be able to ascertain what that change is.

Thus when water freezes we know that the molecules, or smallest parts of the substance, must be arranged differently in ice and in water. We also know that this arrangement in ice must have a certain kind of symmetry, because the ice is in the form of symmetrical crystals, but we have as yet no precise knowledge of the actual arrangement of the molecules in ice. But whenever we can completely describe the change of arrangement we have a knowledge, perfect so far as it extends, of what has taken place, though we may still have to learn the necessary conditions under which a similar event will always take place.

Hence the first part of physical science ralates to the relative position and motion of bodies.

DEFINITION OF A MATERIAL SYSTEM.—In all scientific procedure we begin by marking out a certain region or subject as the field of our investigations. To this we must confine our attention, leaving the rest of the universe out of account till we have completed the investigation in which we are engaged. In physical science, therefore, the first step is to define clearly the material system which we make the subject of our statements. This system may be of any degree of complexity. It may be a single material particle, a body of finite size, or any number of such bodies, and it may even be extended so as to include the whole material universe.

DEFINITION OF INTERNAL AND EXTERNAL.—All relations or actions between one part of this system and another are called Internal relations or actions.

Those between the whole or any part of the system, and bodies not included in the system, are called External rela-

tions or actions. These we study only so far as they affect the system itself, leaving their effect on external bodies out of consideration. Relations and actions between bodies not included in the system are to be left out of consideration. We cannot investigate them except by making our system include these other bodies.

DEFINITION OF CONFIGURATION.--When a material system is considered with respect to the relative position of its parts, the assemblage of relative positions is called the configuration of the system.

A knowledge of the configuration of the system at a given instant implies a knowledge of the position of every point of the system with respect to every other point at that instant.

DIAGRAMS.—The configuration of material systems may be represented in models, plans, or diagrams. The model or diagram is supposed to resemble the

material system only in form, not necessarily in any other respect.

A plan or a map represents on paper in two dimensions what may really be in three dimensions, and can only be completely represented by a model. We shall use the term Diagram to signify any geometrical figure, whether plane or not, by means of which we study the properties of a material system. Thus, when we speak of the configuration of a system, the image which we form in our minds is that of a diagram, which completely represents the configuration, but which has none of the other properties of the material system. Besides diagrams of configuration we may have diagrams of velocity, of stress, &c., which do not represent the form of the system, but by means of which its relative velocities or its internal forces may be studied.

A Material Particle.—A body so small that, *for the purposes of our investigation*, the distances between its

different parts may be neglected, is called a material particle.

Thus in certain astronomical investigations the planets, and even the sun, may be regarded each as a material particle, because the difference of the actions of different parts of these bodies does not come under our notice. But we cannot treat them as material particles when we investigate their rotation. Even an atom, when we consider it as capable of rotation, must be regarded as consisting of many material particles.

The diagram of a material particle is of course a mathematical point, which has no configuration.

Relative Position of two Material Particles.—The diagram of two material particles consists of two points, as, for instance, A and B.

The position of B relative to A is indicated by the direction and length of the straight line $\overline{A\,B}$ drawn *from* A *to* B. If you start from A and travel in the direction indicated by the line $\overline{A\,B}$ and

for a distance equal to the length of that line, you will get to B. This direction and distance may be indicated equally well by any other line, such as $\overline{a\ b}$, which is parallel and equal to $\overline{A\ B}$. The position of A with respect to B is indicated by the direction and length of the line $\overline{B\ A}$, drawn *from* B *to* A, or the line $\overline{b\ a}$, equal and parallel to $\overline{B\ A}$.

It is evident that $\overline{B\ A} = -\overline{A\ B}$.

In naming a line by the letters at its extremities, the order of the letters is always that in which the line is to be drawn.

VECTORS.—The expression $\overline{A\ B}$, in geometry, is merely the name of a line. Here it indicates the operation by which the line is drawn, that of carrying a tracing point in a certain direction for a certain distance. As indicating an operation, $\overline{A\ B}$ is called a Vector, and the operation is completely defined by the direction and distance of the transference. The starting point, which is called the origin of the vector, may be anywhere.

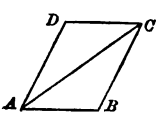
To define a finite straight line we must state its origin as well as its direction and length. All vectors, however, are regarded as equal which are parallel (and drawn towards the same parts) and of the same magnitude.

Any quantity, such, for instance, as a velocity, or a force, which has a definite direction and a definite magnitude may be treated as a vector, and may be indicated in a diagram by a straight line whose direction is parallel to the vector, and whose length represents, according to a determinate scale, the magnitude of the vector.

System of Three Particles.—Let us next consider a system of three particles.

Its configuration is represented by a diagram of three points, A, B, C.

Fig. 1.



The position of B with respect to A is indicated by the vector $\overline{A\,B}$, and that of C with respect to B by the vector $\overline{B\,C}$.

It is manifest that from these data, when A is known, we can find B and then C, so that the configuration of the three points is completely determined.

The position of C with respect to A is indicated by the vector $\overline{A\,C}$, and by the last remark the value of $\overline{A\,C}$ must be deducible from those of $\overline{A\,B}$ and $\overline{B\,C}$.

The result of the operation $\overline{A\,C}$ is to carry the tracing point from A to C. But the result is the same if the tracing point is carried first from A to B and then from B to C, and this is is the sum of the operations $\overline{A\,B}+\overline{B\,C}$.

Addition of Vectors.—Hence the rule for the addition of vectors may be stated thus:—From any point as origin draw the successive vectors in series, so that each vector begins at the end of the preceding one. The straight line from the origin to the extremity of the series represents the vector which is the sum of the vectors.

The order of addition is indifferent, for if we write $\overline{BC}+\overline{AB}$ the operation indicated may be performed by drawing $\overline{AD}$ parallel and equal to $\overline{BC}$, and then joining $\overline{DC}$, which, by Euclid, I. 33, is parallel and equal to $\overline{AB}$, so that by these two operations we arrive at the point C in whichever order we perform them.

The same is true for any number of vectors, take them in what order we please.

SUBTRACTION OF ONE VECTOR FROM ANOTHER.—To express the position of C with respect to B in terms of the positions of B and C with respect to A, we observe that we can get from B to C either by passing along the straight line $\overline{BC}$ or by passing from B to A and then from A to C. Hence

$$\overline{BC}=\overline{BA}+\overline{AC}.$$

$$=\overline{AC}+\overline{BA} \text{ since the order of addition is indifferent.}$$

$$=\overline{AC}-\overline{AB} \text{ since } \overline{AB} \text{ is equal and opposite to } \overline{BA}. \text{ Or, the vec-}$$

tor $\overline{B\,C}$, which expresses the position of C with respect to B, is found by subtracting the vector of B from the vector of C, these vectors being drawn to B and C respectively from any common origin A.

Origin of Vectors.—The positions of any number of particles belonging to a material system may be defined by means of the vectors drawn to each of these particles from some one point. This point is called the origin of the vectors, or, more briefly, the Origin.

This system of vectors determines the configuration of the whole system; for if we wish to know the position of any point B with respect to any other point A, it may be found from the vectors $\overline{O\,A}$ and $\overline{O\,B}$ by the equation

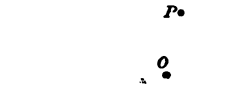
$$\overline{A\,B}=\overline{O\,B}-\overline{O\,A}.$$

We may choose any point whatever for the origin, and there is, for the present, no reason why we should choose one point rather than another. The configuration of the system—that is to say,

the position of its parts with respect to each other—remains the same, whatever point be chosen as origin. Many inquiries, however, are simplified by a proper selection of the origin.

Relative Position of Two Systems. —If the configurations of two different systems are known, each system having its own origin, and if we then wish to include both systems in a larger system, having, say, the same origin as the first of the two systems, we must ascertain the position of the origin of the second system with respect to that of the first, and we must be able to draw lines in the second system parallel to those in the first.

Fig. 2.



Then by "System of Three Particles," the position of a point P of the second

system, with respect to the first origin, O, is represented by the sum of the vector OP of that point with respect to the second origin, O′ and the vector OO′ of the second origin, O′ with respect to the first, O.

THREE DATA FOR THE COMPARISON OF TWO SYSTEMS.—We have an instance of this formation of a large system out of two or more smaller systems, when two neighboring nations, having each surveyed and mapped its own territory, agree to connect their surveys so as to include both countries in one system. For this purpose three things are necessary.

1st. A comparison of the origin selected by the one country with that selected by the other.

2d. A comparison of the directions of reference used in the two countries.

3d. A comparison of the standards of length used in the two countries.

1. In civilized countries latitude is always reckoned from the equator, but

longitude is reckoned from an arbitrary point, as Greenwich or Paris. Therefore, to make the map of Britain fit that of France, we must asertain the difference of longitude between the Observatory of Greenwich and that of Paris.

2. When a survey has been made without astronomical instruments, the directions of reference have sometimes been those given by the magnetic compass. This was, I believe, the case in the original surveys of some of the West India islands. The results of this survey, though giving correctly the local configuration of the island, could not be made to fit properly into a general map of the world till the deviation of the magnet from the true north at the time of the survey was ascertained.

3. To compare the survey of France with that of Britain, the meter, which is the French standard of length, must be compared with the yard, which is the British standard of length.

The yard is defined by Act of Parliament 18 and 19 Vict. c. 72, July 30,

1855, which enacts "that the straight line or distance between the centers of the transverse lines in the two gold plugs in the bronze bar deposited in the office of the Exchequer shall be the genuine standard yard at 62° Fahrenheit, and if lost, it shall be replaced by means of its copies."

The meter derives its authority from a law of the French Republic in 1795. It is defined to be the distance between the ends of a certain rod of platinum made by Borda, the rod being at the temperature of melting ice. It has been found by the measurements of Captain Clarke that the meter is equal to 39.37043 British inches.

On the Idea of Space.—We have now gone through most of the things to be attended to with respect to the configuration of a material system. There remain, however, a few points relating to the metaphysics of the subject, which have a very important bearing on physics.

We have described the method of combining several configurations into one system which includes them all. In this way we add to the small region, which we can explore by stretching our limbs, the more distant regions which we can reach by walking or by being carried. To these we add those of which we learn by the reports of others, and those inaccessible regions whose position we ascertain only by a process of calculation, till at last we recognize that every place has a definite position with respect to every other place, whether the one place is accessible from the other or not.

Thus from measurements made on the earth's surface we deduce the position of the center of the earth relative to known objects, and we calculate the number of cubic miles in the earth's volume quite independently of any hypothesis as to what may exist at the center of the earth, or in any other place beneath that thin layer of the crust of the earth which alone we can directly explore.

Error of Descartes.—It appears then, that the distance between one thing and another does not depend on any material thing between them, as Descartes seems to assert when he says (Princip. Phil., II. 18) that if that which is in a hollow vessel were taken out of it without anything entering to fill its place, the sides of the vessel, having nothing between them, would be in contact.

This assertion is grounded on the dogma of Descartes, that the extension in length, breadth, and depth which constitute space is the sole essential property of matter. "The nature of matter," he tells us, "or of body considered generally, does not consist in a thing being hard, or heavy, or colored, but only in its being extended in length, breadth, and depth" (Princip., II. 4). By thus confounding the properties of matter with those of space he arrives at the logical conclusion, that if the matter within a vessel could be entirely removed the space within the vessel would no

longer exist. In fact he assumes that all space must be always full of matter.

I have referred to this opinion of Descartes in order to show the importance of sound views in elementary dynamics. The primary property of matter was indeed distinctly announced by Descartes in what he calls the "First Law of Nature" (Princip., II. 37): "That every individual thing, so far as in it lies, perseveres in the same state, whether of motion or of rest."

We shall see when we come to Newton's laws of motion that in the words "so far as in it lies," properly understood, is to be found the true primary definition of matter, and the true measure of its quantity. Descartes, however, never attained to a full understanding of his own words (*quantum in se est*), and so fell back on his original confusion of matter with space—space being, according to him, the only form of substance, and all existing things but affections of space. This error runs through every part of Descartes' great work, and it

forms one of the ultimate foundations of the system of Spinoza. I shall not attempt to trace it down to more modern times, but I would advise those who study any system of metaphysics to examine carefully that part of it which deals with physical ideas.

We shall find it more conducive to scientific progress to recognise, with Newton, the ideas of time and space as distinct, at least in thought, from that of the material system whose relations these ideas serve to co-ordinate.

On the Idea of Time.—The idea of Time in its most primitive form is probably the recognition of an order of sequence in our states of consciousness. If my memory were perfect, I might be able to refer every event within my own experience to its proper place in a chronological series. But it would be difficult, if not impossible, for me to compare the interval between one pair of events and that between another pair —to ascertain, for instance, whether the

time during which I can work without feeling tired is greater or less now than when I first began to study. By our intercourse with other persons, and by our experience of natural processes which go on in a uniform or a rhythmical manner, we come to recognize the possibility of arranging a system of chronology in which all events whatever, whether relating to ourselves or to others, must find their place. Of any two events, say the actual disturbance at the star in Corona Borealis, which caused the luminous effects examined spectroscopically by Mr. Huggins on the 16th May, 1866, and the mental suggestion which first led Professor Adams or M. Leverrier to begin the researches which led to the discovery, by Dr. Galle, on the 23rd September, 1846, of the planet Neptune, the first named must have occured either before or after the other, or else at the same time.

Absolute, true, and mathematical Time is conceived by Newton as flowing at a constant rate, unaffected by the speed or

slowness of the motions of material things. It is also called duration. Relative, apparent, and common time is duration as estimated by the motion of bodies, as by days, months, and years. These measures of time may be regarded as provisional, for the progress of astronomy has taught us to measure the inequality in the lengths of days, months, and years, and thereby to reduce the apparent time to a more uniform scale, called Mean Solar Time.

Absolute Space.—Absolute space is conceived as remaining always similar to itself and immovable. The arrangement of the parts of space can no more be altered than the order of the portions of time. To conceive them to move from their places is to conceive a place to move away from itself.

But as there is nothing to distinguish one portion of time from another except by the different events which occur in them, so there is nothing to distinguish one part of space from another except its

relation to the place of material bodies. We cannot describe the time of an event except by reference to some other event, or the place of a body except by reference to some other body. All our knowledge, both of time and place, is essentially relative. When a man has acquired the habit of putting words together, without troubling himself to form the thoughts which ought to correspond to them, it is easy for him to frame an antithesis between this relative knowledge and a so-called absolute knowlege, and to point out our ignorance of the absolute position of a point as an instance of the limitation of our faculties. Any one, however, who will try to imagine the state of a mind conscious of knowing the absolute position of a point will ever after be content with our relative knowledge.

Statement of the General Maxim of Physical Science.—There is a maxim which is often quoted, that "The same causes will always produce the same effects."

To make this maxim intelligible we must define what we mean by the same causes and the same effects, since it is manifest that no event ever happens more than once, so that the causes and effects cannot be the same in *all* respects. What is really meant is that if the causes differ only as regards the absolute time or the absolute place at which the event occurs, so likewise will the effects.

The following statement, which is equivalent to the above maxim, appears to be more definite, more explicitly connected with the ideas of space and time, and more capable of application to particular cases:

"The difference between one event and another does not depend on the mere difference of the times or the places at which they occur, but only on differences in the nature, configuration, or motion of the bodies concerned."

It follows from this, that if an event has occurred at a given time and place

it is possible for an event exactly similar to occur at any other time and place.

There is another maxim which must not be confounded with that quoted at the beginning of this article, which asserts "That like causes produce like effects."

This is only true when small variations in the initial circumstances produce only small variations in the final state of the system. In a great many physical phenomena this condition is satisfied; but there are other cases in which a small initial variation may produce a very great change in the final state of the system, as when the displacement of the points causes a railway train to run into another instead of keeping its proper course.

CHAPTER II.

ON MOTION.

DEFINITION OF DISPLACEMENT.—We have already compared the position of different points of a system at the same

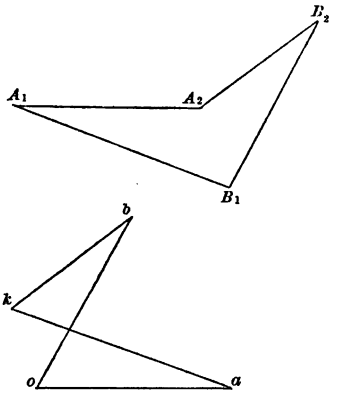
instant of time. We have next to compare the position of a point at a given instant with its position at a former instant, called the Epoch.

The vector which indicates the final position of a point with respect to its position at the epoch is called the Displacement of that point. Thus if A_1 is the initial and A_2 the final position of the point A, the line $\overline{A_1 A_2}$ is the displacement of A, and any vector $\overline{o a}$ drawn from the origin o parallel and equal to $\overline{A_1 A_2}$ indicates this displacement.

Diagram of Displacement. — If another point of the system is displaced from B_1 to B_2 the vector $\overline{o b}$ parallel and equal to $\overline{B_1 B_2}$ indicates the displacement of B.

In like manner the displacement of any number of points may be represented by vectors drawn from the same origin o. This system of vectors is called the Diagram of Displacement. It is not necessary to draw actual lines to repre-

FIG. 3.



sent these vectors; it is sufficient to indicate the points a, b, &c., at the extremities of the vectors. The diagram of displacement may therefore be regarded as consisting of a number of points, a, b, &c., corresponding with the material particles, A, B, &c., belonging to the system, together with a point o, the position of which is arbitrary, and which is the assumed origin of all the vectors.

RELATIVE DISPLACEMENT.—The line $\overline{a\ b}$ in the diagram of displacement represents the displacement of the point B with respect to A.

For if in the diagram of displacement (Fig. 3) we draw $\overline{a\ k}$ parallel and equal to $\overline{B_1\ A_1}$, and in the same direction, and join $\overline{k\ b}$, it is easy to show that $\overline{k\ b}$ is equal and parallel to $\overline{A_2\ B_2}$.

For the vector $\overline{k\ b}$ is the sum of the vectors $\overline{k\ a}$, $\overline{a\ o}$, and $\overline{o\ b}$, and $\overline{A_2\ B_2}$ is the sum of $\overline{A_2\ A_1}$, $\overline{A_1 B_1}$, and $\overline{B_1\ B_2}$. But of these $\overline{k\ a}$ is the same as $\overline{A_1\ B_2}$, $\overline{o\ o}$ is the same as $\overline{A_2\ A_1}$, and $\overline{o\ b}$ is the same as $B_1\ B_2$, and by "Addition of Vectors," the order of summation is indifferent, so that the vector $\overline{k\ b}$ is the same, in direction and magnitude, as $\overline{A_2\ B_2}$. Now $\overline{k\ a}$, or $\overline{A_1\ B_1}$ represents the original position of B with respect to A, and $\overline{k\ b}$, or $\overline{A_2\ B_2}$ represents the final position of B with respect to A. Hence $\overline{a\ b}$ represents the displacement of B with respect to A, which was to be proved.

In "Definition of Displacement," we purposely omitted to say whether the

origin to which the original configuration was referred, and that to which the final configuration is referred, are absolutely the same point, or whether, during the displacement of the system, the origin also is displaced.

We may now, for the sake of argument, suppose that the origin is absolutely fixed, and that the displacements represented by $\overline{o\,a}$, $\overline{o\,b}$, &c., are the absolute displacements. To pass from this case to that in which the origin is displaced we have only to take A, one of the movable points, as origin. The absolute displacement of A being represented by $\overline{o\,a}$, the displacement of B with respect to A is represented, as we have seen, by $\overline{a\,b}$, and so on for any other points of the system.

The arrangement of the points a, b, &c., in the diagram of displacement is therefore the same, whether we reckon the displacements with respect to a fixed point or a displaced point; the only difference is that we adopt a different origin of vectors in the diagram of displacements,

the rule being that whatever point we take, whether fixed or moving, for the origin of the diagram of configuration, we take the corresponding point as origin in the diagram of displacement. If we wish to indicate the fact that we are entirely ignorant of the absolute displacement in space of any point of the system we may do so by construeting the diagram of displacements as a mere system of points, without indicating in any way which of them we take as the origin.

This diagram of displacements (without an origin) will then represent neither more nor less than all we can ever know about the displacement of the system. It consists simply of a number of points, a, b, c, &c., corresponding to the points A, B, C, of the material system, and a vector, as $\overline{a\,b}$ represents the displacement of B with respect to A.

UNIFORM* DISPLACEMENT.—When the displacements of all points of a material

* When the simultaneous values of a quantity for different bodies or places are equal, the quantity is said to be *uniformly* distributed in space.

system with respect to an external point are the same in direction and magnitude, the diagram of displacement is reduced to two points—one corresponding to the external point, and the other to each and every point of the displaced system. In this case the points of the system are not displaced with respect to one another, but only with respect to the external point.

This is the kind of displacement which occurs when a body of invariable form moves parallel to itself. It may be called uniform displacement.

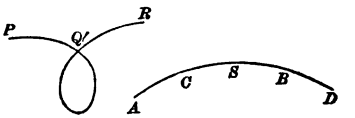
On Motion.—When the change of configuration of a system is considered with respect only to its state at the beginning and the end of the process of change, and without reference to the time during which it takes place, it is called the displacement of the system.

When we turn our attention to the process of change itself, as taking place during a certain time and in a continuous manner, the change of configura-

tion is ascribed to the motion of the system.

On the Continuity of Motion.—When a material particle is displaced so as to pass from one position to another, it can only do so by traveling along some course or path from the one position to the other.

Fig. 4.



At any instant during the motion the particle will be found at some one point of the path, and if we select any point of the path, the particle will pass that point once at least* during its motion.

This is what is meant by saying that the particle described a continuous path.

* If the path cuts itself so as to form a loop, as P, Q, R, (fig. 4), the particle will pass the point of intersection, Q, twice, and if the particle returns on its own path, as in the path A, B, C, D, it may pass the same point, S, three or more times.

The motion of a material particle which has continuous existence in time and space is the type and exemplar of every form of continuity.

On Constant* Velocity.—If the motion of a particle is such that in equal intervals of time, however short, the displacements of the particle are equal and in the same direction, the particle is said to move with constant velocity.

It is manifest that in this case the path of the body will be a straight line, and the length of any part of the path will be proportional to the time of describing it.

The rate or speed of the motion is called the velocity of the particle, and its magnitude is expressed by saying that it is such a distance in such a time, as, for instance, ten miles an hour, or one meter per second. In general we select a unit of time, such as a second and measure

* When the successive values of a quantity for successive instances of time are equal, the quantity is said to be *constant*.

velocity by the distance described in unit of time.

If one meter be described in a second and if the velocity be constant, a thousandth or a millionth of a meter will be described in a thousandth or a millionth of a second. Hence, if we can observe or calculate the displacement during any interval of time, however short, we may deduce the distance which would be described in a longer time with the same velocity. This result, which enables us to state the velocity during the short interval of time, does not depend on the body's actually continuing to move at the same rate during the longer time. Thus we may know that a body is moving at the rate of ten miles an hour, though its motion at this rate may last for only the hundredth of a second.

On the Measurement of Velocity when Variable.—When the velocity of a particle is not constant, its value at any given instant is measured by the distance which would be described in

unit of time by a body having the same velocity as that which the particle has at that instant.

Thus when we say that at a given instant, say one second after a body has begun to fall, its velocity is 980′ centimeters per second, we mean that if the velocity of a particle were constant and equal to that of the falling body at the given instant, it would describe 980 centimeters in a second.

It is specially important to understand what is meant by the velocity or rate of motion of a body, because the ideas which are suggested to our minds by considering the motion of a particle are those which Newton made use of in his method of Fluxions,* and they lie at the foundation of the great extension of exact science which has taken place in modern times.

* According to the method of Fluxions, when the value of one quantity depends on that of another, the rate of variation of the first quantity with respect to the second may be expressed as a velocity, by imagining the first quantity to represent the displacement of a particle, while the second flows uniformly with the time.

DIAGRAM OF VELOCITIES.—If the velocity of each of the bodies in the system is constant, and if we compare the configurations of the system at an interval of a unit of time, then the displacements, being those produced in unit of time in bodies moving with constant velocities, will represent those velocities according to the method of measurement described "On Constant Velocity."

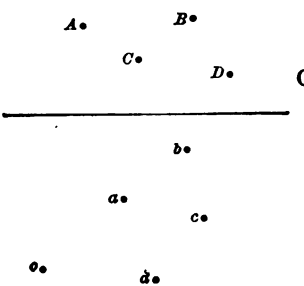
If the velocities do not actually continue constant for a unit of time, then we must imagine another system consisting of the same number of bodies, and in which the velocities are the same as those of the corresponding bodies of the system at the given instant, but remain constant for a unit of time. The displacements of this system represent the velocities of the actual system at the given instant.

Another mode of obtaining the diagram of velocities of a system at a given instant is to take a small interval of time, say the nth part of the unit of time, so that the middle of this interval

corresponds to the given instant. Take the diagram of displacements corresponding to this interval and magnify all its dimensions n times. The result will be a diagram of the *mean* velocities of the system during the interval. If we now suppose the number n to increase without limit the interval will diminish without limit, and the mean velocities will approximate without limit to the actual velocities at the given instant. Finally, when n becomes infinite the diagram will represent accurately the velocities at the given instant.

PROPERTIES OF THE DIAGRAM OF VELOCITIES. (Fig. 5).—The diagram of

FIG. 5.



velocities for a system consisting of a number of material particles consists of a number of points, each corresponding to one of the particles.

The velocity of any particle B with respect to any other, A, is represented in direction and magnitude by the line $\overline{a\,b}$ in the diagram of velocities, drawn from the point a, corresponding to A, to the point b, corresponding to B.

We may in this way find, by means of the diagram, the relative velocity of any two particles. The diagram tells us nothing about the absolute velocity of any point; it expresses exactly what we can know about the motion and no more. If we choose to imagine that $\overline{oa}$ represents the absolute velocity of A, then the absolute velocity of any other particle, B, will be represented by the vector $\overline{o\,b}$, drawn from o as origin, to the point b, which corresponds to B.

But as it is impossible to define the position of a body except with respect to the position of some point of reference, so it is impossible to define the

velocity of a body, except with respect to the velocity of the point of reference. The phrase absolute velocity has as little meaning as absolute position. It is better, therefore, not to distinguish any point in the diagram of velocity as the origin, but to regard the diagram as expressing the relations of all the velocities without defining the absolute value of any one of them.

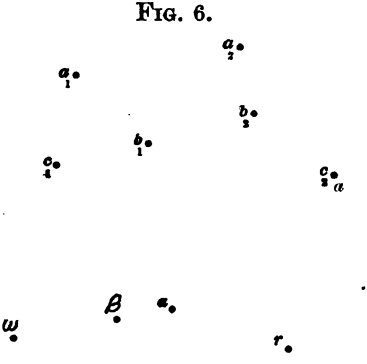
Meaning of the Phrase "At Rest." —It is true that when we say that a body is at rest we use a form of words which appears to assert something about that body considered in itself, and we might imagine that the velocity of another body, if reckoned with respect to a body at rest, would be its true and only absolute velocity. But the phrase "at rest" means in ordinary language "having no velocity with respect to that on which the body stands," as, for instance, the surface of the earth or the deck of a ship. It cannot be made to mean more than this.

It is therefore unscientific to distinguish between rest and motion, as between two different states of a body in itself, since it is impossible to speak of a body being at rest or in motion except with reference, expressed or implied, to some other body.

On Change of Velocity.—As we have compared the velocities of different bodies at the same time, so we may compare the relative velocity of one body with respect to another at different times.

If a_1 b_1 c_1, be the diagram of the veloc-

Fig. 6.



ities of the system of bodies, A, B, C, in its original state, and if a_2, b_2, c_2, be the diagram of velocities in the final state of the system, then if we take any point ω as origin and draw $\overline{\omega\, a}$ equal and parallel to $\overline{a_1\, a_2}$, $\omega\, \beta$ equal and parallel to $\overline{b_1\, b_2}$, $\omega\, \gamma$ equal and parallel to $\overline{c_1\, c_2}$, and so on, we shall form a diagram of points a, β, γ, &c., such that any line $\overline{a\, \beta}$ in this diagram represents in direction and magnitude the change of the velocity of B with respect to A. This diagram may be called the diagram of Total Accelerations.

On Acceleration.—The word Acceleration is here used to denote any change in the velocity, whether that change be an increase, a diminution, or a change of direction. Hence, instead of distinguishing, as in ordinary language, between the acceleration, the retardation, and the deflection of the motion of a body, we say that the acceleration may be in the direction of motion, in the contrary direction, or transverse to that direction.

As the displacement of a system is defined to be the change of the configura tion of the system, so the Total Acceleration of the system is defined to be the change of the velocities of the system. The process of constructing the diagram of total accelerations, by a comparison of the initial and final diagrams of velocities, is the same as that by which the diagram of displacements was constructed by a comparison of the initial and final diagrams of configuration.

On the Rate of Acceleration.—We have hitherto been considering the total acceleration which takes place during a certain interval of time. If the rate of acceleration is constant, it is measured by the total acceleration in a unit of time. If the rate of acceleration is variable, its value at a given instant is measured by the total acceleration in unit of time of a point whose acceleration is constant and equal to that of the particle at the given instant.

It appears from this definition that

the method of deducing the rate of acceleration from a knowledge of the total acceleration in any given time is precisely analogous to that by which the velocity at any instant is deduced from a knowledge of the displacement in any given time.

The diagram of total accelerations constructed for an interval of the nth part of the unit of time, and then magnified n times, is a diagram of the mean rates of acceleration during that interval, and by taking the interval smaller and smaller, we ultimately arrive at the true rate of acceleration at the middle of that interval.

As rates of acceleration have to be considered in physical science much more frequently than total accelerations, the word acceleration has come to be employed in the sense in which we have hitherto used the phrase—rate of acceleration.

In future, therefore, when we use the word acceleration without qualification, we mean what we have here described as the rate of acceleration.

DIAGRAM OF ACCELERATIONS.—The diagram of accelerations is a system of points, each of which corresponds to one of the bodies of the material system, and is such that any line $\overline{\alpha\beta}$ in the diagram represents the rate of acceleration of the body B with respect to the body A.

It may be well to observe here that in the diagram of configuration we use the capital letters, A, B, C, &c., to indicate the relative position of the bodies of the system; in the diagram of velocities we use the small letters, a, b, c, to indicate the relative velocities of these bodies; and in the diagram of accelerations we use the Greek letters, α, β, γ, to indicate their relative accelerations.

ACCELERATION A RELATIVE TERM.—Acceleration, like position and velocity, is a relative term and cannot be interpreted absolutely.

If every particle of the material universe within the reach of our means of observation were at a given instant to have its velocity altered by compound-

ing therewith a new velocity, the same in magnitude and direction for every such particle, all the relative motions of bodies within the system would go on in a perfectly continuous manner, and neither astronomers nor physicists, though using their instruments all the while, would be able to find out that anything had happened.

It is only if the change of motion occurs in a different manner in the different bodies of the system that any event capable of being observed takes place.

CHAPTER III.

ON FORCE.

KINEMATICS AND KINETICS.—We have hitherto been considering the motion of a system in its purely geometrical aspect. We have shown how to study and describe the motion of such a system, howevery arbitrary, without taking into account any of the conditions of motion which arise from the mutual action between the bodies.

The theory of motion treated in this way is called Kinematics. When the mutual action between bodies is taken into account, the science of motion is called Kinetics, and when special attention is paid to force as the cause of motion, it is called Dynamics.

MUTUAL ACTION BETWEEN TWO BODIES —STRESS.—The mutual action between two portions of matter receives different names according to the aspect under which it is studied, and this aspect depends on the extent of the material system which forms the subject of our attention.

If we take into account the whole phenomenon of the action between the two portions of matter, we call it Stress. This stress, according to the mode in which it acts, may be described as Attraction, Repulsion, Tension, Pressure, Shearing stress, Torsion, &c.

EXTERNAL FORCE.—But if, as in "Definition of a Material System," we

confine our attention to one of the portions of matter, we see, as it were, only one side of the transaction—namely, that which affects the portion of matter under our consideration—and we call this aspect of the phenomenon, with reference to its effect, an External Force acting on that portion of matter, and with reference to its cause we call the Action of the other portion of matter. The opposite aspect of the stress is called the Reaction on the other portion of matter.

DIFFERENT ASPECTS OF THE SAME PHENOMENON.—In commercial affairs the same transaction between two parties is called Buying when we consider one party, Selling when we consider the other, and Trade when we take both parties into consideration.

The accountant who examines the records of the transaction finds that the two parties have entered it on opposite sides of their respective ledgers, and in comparing the books he must in every

case bear in mind in whose interest each book is made up.

For similar reasons in dynamical investigations we must always remember which of the two bodies we are dealing with, so that we may state the forces in the interest of that body, and not set down any of the forces on the wrong side of the account.

NEWTON'S LAWS OF MOTION.—External or "impressed" force considered with reference to its effect—namely, the alteration of the motions of bodies—is completely defined and described in Newton's three laws of motion.

The first law tells us under what conditions there is no external force.

The second shows us how to measure the force when it exists.

The third compares the two aspects of the action between two bodies, as it affects the one body or the other.

THE FIRST LAW OF MOTION.—LAW I. *Every body perseveres in its state of rest*

or of moving uniformly in a straight line, except in so far as it is made to change that state by external forces.

The experimental argument for the truth of this law is, that in every case in which we find an alteration of the state of motion of a body, we can trace this alteration to some action between that body and another, that is to say, to an external force. The existence of this action is indicated by its effects on the other body when the motion of that body can be observed. Thus the motion of a cannon ball is retarded, but this arises from an action between the projectile and the air which surrounds it, whereby the ball experiences a force in the direction opposite to its relative motion, while the air, pushed forward by an equal force, is itself set in motion, and constitutes what is called the *wind* of the cannon ball.

But our conviction of the truth of this law may be greatly strengthened by considering what is involved in a denial of it. Given a body in motion. At a

given instant let it be left to itself and not acted on by any force. What will happen? According to Newton's law it will persevere in moving uniformly in a straight line, that is, its velocity will remain constant both in direction and magnitude.

If the velocity does not remain constant let us suppose it to vary. The change of velocity, as we saw "On Change of Velocity," must have a definite direction and magnitude. By the "Statement of the General Maxim of Physical Science," this variation must be the same whatever be the time or place of the experiment. The direction of the change of motion must therefore be determined either by the direction of the motion itself, or by some direction fixed in the body.

Let us, in the first place, suppose the law to be that the velocity diminishes at a certain rate, which for the sake of the argument we may suppose so slow that by no experiments on moving bodies could we have detected the diminution of velocity in hundreds of years.

The velocity referred to in this hypothetical law can only be the velocity referred to a point absolutely at rest. For if it is a relative velocity its direction as well as its magnitude depends on the velocity of the point of reference.

If, when referred to a certain point, the body appears to be moving northward with diminishing velocity, we have only to refer it to another point moving northward with a uniform velocity greater than that of the body, and it will apper to be moving southward with increasing velocity.

Hence the hypothetical law is without meaning, unless we admit the possibility of defining absolute rest and absolute velocity.

Even if we admit this as a possibility, the hypothetical law, if found to be true, might be interpreted, not as a contradiction of Newton's law, but as evidence of the resisting action of some medium in space.

To take another case. Suppose the law to be that a body, not acted on by

any force, ceases at once to move. This is not only contradicted by experience, but it leads to a definition of absolute rest as the state which a body assumes as soon as it is freed from the action of external forces.

It may thus be shown that the denial of Newton's law is in contradiction to the only system of consistent doctrine about space and time which the human mind has been able to form.

On the Equilibrium of Forces.—If a body moves with constant velocity in a straight line, the external forces, if any, which act on it, balance each other, or are in equilibrium.

Thus if a carriage in a railway train moves with constant velocity in a straight line, the external forces which act on it —such as the traction of the carriage in front of it pulling it forwards, the drag of that behind it, the friction of the rails, the resistance of the air acting backwards, the weight of the carriage acting downwards, and the pressure of

the rails acting upwards—must exactly balance each other.

Bodies at rest with respect to the surface of the earth are really in motion, and their motion is not constant nor in a straight line. Hence the forces which act on them are not exactly balanced. The apparent weight of bodies is estimated by the upward force required to keep them at rest relatively to the earth. The apparent weight is therefore rather less than the attraction of the earth, and makes a smaller angle with the axis of the earth, so that the combined effect of the supporting force and the earth's attraction is a force perpendicular to the earth's axis just sufficient to cause the body to keep to the circular path which it must describe if resting on the earth.

Definition of Equal Times.—The first law of motion, by stating under what circumstances the velocity of a moving body remains constant, supplies us with a method of defining equal intervals of time. Let the material system

consist of two bodies which do not act on one another, and which are not acted on by any body external to the system. If one of these bodies is in motion with respect to the other, the relative velocity will, by the first law of motion, be constant and in a straight line.

Hence intervals of time are equal when the relative displacements during those intervals are equal.

This might at first sight appear to be nothing more than a definition of what we mean by equal intervals of time, an expression which we have not hitherto defined at all.

But if we suppose another moving system of two bodies to exist, each of which is not acted upon by any body whatever, this second system will give us an independent method of comparing intervals of time.

The statement that equal intervals of time are those during which equal displacements occur in any such system, is therefore equivalent to the assertion that the comparison of intervals of time leads

to the same result, whether we use the first system of two bodies or the second system as our time-piece.

We thus see the theoretical possibility of comparing intervals of time however distant, though it is hardly necessary to remark that the method cannot be put in practice in the neighborhood of the earth, or any other large mass of gravitating matter.

THE SECOND LAW OF MOTION.—LAW II.—*Change of motion is proportional to the impressed force, and takes place in the direction in which the force is impressed.*

By motion Newton means what in modern scientific language is called momentum, in which the quantity of matter moved is taken into account as well as the rate at which it travels.

By impressed force he means what is now called Impulse, in which the time during which the force acts is taken into account as well as the intensity of the force.

DEFINITION OF EQUAL MASSES AND OF EQUAL FORCES.—An exposition of the law therefore involves a definition of equal quantities of matter and of equal forces.

We shall assume that it is possible to cause the force with which one body acts on another to be of the same intensity on different occasions.

If we admit the permanency of the properties of bodies this can be done. We know that a thread of caoutchouc when stretched beyond a certain length exerts a tension which increases the more the thread is elongated. On account of this property the thread is said to be elastic. When the same thread is drawn out to the same length it will, if its properties remain constant, exert the same tension. Now let one end of the thread be fastened to a body, M, not acted on by any other force than the tension of the thread, and let the other end be held in the hand and pulled in a constant direction with a force just sufficient to elongate the thread to a given length; the force act-

ing on the body will then be of a given intensity, F. The body will acquire velocity, and at the end of a unit of time this velocity will have a certain value, V.

If the same string be fastened to another body, N, and pulled as in the former case, so that the elongation is the same as before, the force acting on the body will be the same, and if the velocity communicated to N in a unit of time is also the same, namely, V, then we say of the two bodies M and N that they consist of equal quantities of matter, or, in modern language, they are equal in mass. In this way, by the use of an elastic string, we might adjust the masses of a number of bodies so as to be each equal to a standard unit of mass, such as a pound avoirdupois, which is the standard of mass in Britain.

Measurement of Mass.—The scientific value of the dynamical method of comparing quantities of matter is best seen by comparing it with other methods in actual use.

As long as we have to do with bodies of exactly the same kind, there is no difficulty in understanding how the quantity of matter is to be measured. If equal quantities of the substance produce equal effects of any kind, we may employ these effects as measures of the quantity of the substance.

For instance, if we are dealing with sulphuric acid of uniform strength, we may estimate the quantity of a given portion of it in several different ways. We may weigh it, we may pour it into a graduated vessel, and so measure its volume, or we may ascertain how much of a standard solution of potash it will neutralize.

We might use the same methods to estimate a quantity of nitric acid if we were dealing only with nitric acid; but if we wished to compare a quantity of nitric acid with a quantity of sulphuric acid we should obtain different results by weighing, by measuring, and by testing with an alkaline solution.

Of these three methods, that of weigh-

ing depends on the attraction between the acid and the earth, that of measuring depends on the volume which the acid occupies, and that of titration depends on its power of combining with potash.

In abstract dynamics, however, matter is considered under no other aspect than as that which can have its motion changed by the application of force. Hence any two bodies are of equal mass if equal forces applied to these bodies produce, in equal times, equal changes of velocity. This is the only definition of equal masses which can be admitted in dynamics, and it is applicable to all material bodies, whatever they may be made of.

It is an observed fact that bodies of equal mass, placed in the same position relative to the earth, are attracted equally towards the earth, whatever they are made of; but this is not a doctrine of abstract dynamics, founded on axiomatic principles, but a fact discovered by observation, and verified by the careful experiments of Newton,* on the times of

* "Principia," III., Prop. 6.

oscillation of hollow wooden balls suspended by strings of the same length, and containing gold, silver, lead, glass, sand, common salt, wood, water, and wheat.

The fact, however, that in the same geographical position the weights of equal masses are equal, is so well established, that no other mode of comparing masses than that of comparing their weights is ever made use of, either in commerce or in science, except in researches undertaken for the special purpose of determining, in absolute measure, the weight of unit of mass at different parts of the earth's surface. The method employed in these researches is essentially the same as that of Newton, namely, by measuring the length of a pendulum which swings seconds.

The unit of mass in this country is defined by the Act of Parliament (18 & 19 Vict. c. 72, July 30, 1855) to be a piece of platinum marked "P. S., 1844, 1 lb." deposited in the office of the Exchequer, which "shall be and be de-

nominated the Imperial Standard Pound Avoirdupois." One seven-thousandth part of this pound is a grain. The French Standard of mass is the "Kilogramme des Archives," made of platinum by Borda. Professor Miller finds the kilogramme equal to 15432.34874 grains.

NUMERICAL MEASUREMENT OF FORCE. —The unit of force is that force which, acting on the unit of mass for the unit of time, generates unit of velocity.

Thus the weight of a gramme—that is to say, the force which causes it to fall —may be asserted by letting it fall freely. At the end of one second its velocity will be about 981 centimeters per second if the experiment be in Britain. Hence the weight of a gramme is represented by the number 981, if the centimeter, the gramme, and the second are taken as the fundamental units.

It is sometimes convenient to compare forces with the weight of a body, and to speak of a force of so many pounds weight or grammes weight. This is

called Gravitation measure. We must remember, however, that though a pound or a gramme is the same all over the world, the weight of a pound or a gramme is greater in high latitudes than near the equator, and therefore a measurement of force in gravitation measure is of no scientific value unless it is stated in what part of the world the measurement was made.

If, as in Britain, the units of length, mass, and time are one foot, one pound, and one second, the unit of force is that which, in one second, would communicate to one pound a velocity of one foot per second. This unit of force is called a *Poundal.*

In the French metric system the units are one centimeter, one gramme, and one second. The force which in one second would communicate to one gramme a velocity of one centimeter per second is called a *Dyne.*

Since the foot is 30.4797 centimeters and the pound is 453.59 grammes, the poundal is 13825.38 dynes.

SIMULTANEOUS ACTION OF FORCES ON A BODY.—Now let a unit of force act for unit of time upon unit of mass. The velocity of the mass will be changed, and the total acceleration will be unity in the direction of the force.

The magnitude and direction of this total acceleration will be the same whether the body is originally at rest or in motion. For the expression "at rest" has no scientific meaning, and the expression "in motion," if it refers to relative motion, may mean anything, and if it refers to absolute motion can only refer to some medium fixed in space. To discover the existence of a medium, and to determine our velocity with respect to it by observation on the motion of bodies, is a legitimate scientific inquiry, but supposing all this done we should have discovered, not an error in the laws of motion, but a new fact in science.

Hence the effect of a given force on a body does not depend on the motion of that body.

Neither is it affected by the simulta-

neous action of other forces on the body. For the effect of these forces on the body is only to produce motion in the body, and this does not affect the acceleration produced by the first force.

Hence we arrive at the following form of the law. *When any number of forces act on a body, the acceleration due to each force is the same in direction and magnitude as if the others had not been in action.*

When a force, constant in direction and magnitude, acts on a body, the total acceleration is proportional to the interval of time during which the force acts.

For if the force produces a certain total acceleration in a given interval of time, it will produce an equal total acceleration in the next, because the effect of the force does not depend upon the velocity which the body has when the force acts on it. Hence in every equal interval of time there will be an equal change of the velocity, and the total change of velocity from the beginning of the motion will be proportional to the time of action of the force.

The total acceleration in a given time is proportional to the force.

For if several equal forces act in the same direction on the same body in the same direction, each produces its effect independently of the others. Hence the total acceleration is proportional to the number of the equal forces.

On Impulse.—The total effect of a force in communicating velocity to a body is therefore proportional to the force and to the time during which it acts conjointly.

The product of the time of action of a force into its intensity if it is constant, or its mean intensity if it is variable, is called the *Impulse* of the force.

There are certain cases in which a force acts for so short a time that it is difficult to estimate either its intensity or the time during which it acts. But it is comparatively easy to measure the effect of the force in altering the motion of the body on which it acts, which, as we have seen, depends on the impulse.

The word impulse was originally used to denote the effect of a force of short duration, such as that of a hammer striking a nail. There is no essential difference, however, between this case and any other case of the action of force. We shall therefore use the word impulse as above defined, without restricting it to cases in which the action is of an exceptionally transient character.

Relation between Force and Mass. —If a force acts on a unit of mass for a certain interval of time, the impulse, as we have seen, is measured by the velocity generated.

If a number of equal forces act in the same direction, each on a unit of mass, the different masses will all move in the same manner, and may be joined together into one body without altering the phenomenon. The velocity of the whole body is equal to that produced by one of the forces acting on a unit of mass.

Hence the force required to produce a

given change of velocity in a given time is proportional to the number of units of mass of which the body consists.

On Momentum.—The numerical value of the Momentum of a body is the product of the number of units of mass in the body into the number of units of velocity with which it is moving.

The momentum of any body is thus measured in terms of the momentum of unit of mass moving with unit of velocity, which is taken as the unit of momentum.

The direction of the momentum is the same as that of the velocity, and as the velocity can only be estimated with respect to some point of reference, so the particular value of the momentum depends on the point of reference which we assume. The momentum of the moon, for example, will be very different according as we take the earth or the sun for the point of reference.

Statement of the Second Law of Motion in Terms of Impulse and Mo-

MENTUM.—*The change of momentum of a body is numerically equal to the impulse which produces it, and is in the same direction.*

ADDITION OF FORCES.—If any number of forces act simultaneously on a body, each force produces an acceleration proportional to its own magnitude ("Measurement of Mass"). Hence if in the diagram of accelerations (See "Diagram of Accelerations") we draw from any origin a line representing in direction and magnitude the acceleration due to one of the forces, and from the end of this line another representing the acceleration due to another force, and so on, drawing lines for each of the forces taken in any order, then the line drawn from the origin to the extremity of the last of the lines will represent the acceleration due to the combined action of all the forces.

Since in this diagram lines which represent the accelerations are in the same proportion as the forces to which these accelerations are due, we may consider

the lines as representing these forces themselves. The diagram, thus understood, may be called a Diagram of Forces, and the line from the origin to the extremity of the series represents the Resultant Force.

An important case is that in which the set of lines representing the forces terminate at the origin so as to form a closed figure. In this case there is no resultant force, and no acceleration. The effects of the forces are exactly balanced, and the case is one of equilibrium. The discussion of cases of equilibrium forms the subject of the science of Statics.

. It is manifest that since the system of forces is exactly balanced, and is equivalent to no force at all, the forces will also be balanced if they act in the same way on any other material system, whatever be the mass of that system. This is the reason why the consideration of mass does not enter into statical investigations.

THE THIRD LAW OF MOTION.—LAW III.—*Reaction is always equal and opposite to action, that is to say, the actions of two bodies upon each other are always equal and in opposite directions.*

When the bodies between which the action takes place are not acted on by any other force, the changes in their respective momenta produced by the action are equal and in opposite directions.

The changes in the velocities of the two bodies are also in opposite directions, but not equal, except in the case of equal masses. In other cases the changes of velocity are in the inverse ratio of the masses.

ACTION AND REACTION ARE THE PARTIAL ASPECTS OF A STRESS.—We have already ("Mutual Action—Stress," etc.) used the word stress to denote the mutual action between two portions of matter. This word was borrowed from common language, and invested with a precise scientific meaning by the late Professor Rankine, to whom we are indebted for several other valuable scientific terms.

As soon as we have formed for ourselves the idea of a stress, such as the Tension of a rope or the Pressure between two bodies, and have recognized its double aspect as it affects the two portions of matter between which it acts, the third law of motion is seen to be equivalent to the statement that all force is of the nature of stress, that stress exists only between two portions of matter, and that its effects on these portions of matter (measured by the momentum generated in a given time) are equal and opposite.

The stress is measured numerically by the force exerted on either of the two portions of matter. It is distinguished as a tension when the force acting on either portion is towards the other, and as a pressure when the force acting on either portion is away from the other.

When the force is inclined to the surface which separates the two portions of matter the stress cannot be distinguished by any term in ordinary language, but must be defined by technical mathematical terms.

When a tension is exerted between two bodies by the medium of a string, the stress, properly speaking, is between any two parts into which the string may be supposed to be divided by an imaginary section or transverse interface. If, however, we neglect the weight of the string, each portion of the string is in equilibrium under the action of the tensions at its extremities, so that the tensions at any two transverse interfaces of the string must be the same. For this reason we often speak of the tension of the string as a whole, without specifying any particular section of it, and also the tension between the two bodies, without considering the nature of the string through which the tension is exerted.

Attraction and Repulsion.—There are other cases in which two bodies at a distance appear mutually to act on each other, though we are not able to detect any intermediate body, like the string in the former example, through which the action takes place. For instance,

two magnets or two electrified bodies appear to act on each other when placed at considerable distances apart, and the motions of the heavenly bodies are observed to be affected in a manner which depends on their relative position.

This mutual action between distant bodies is called attraction when it tends to bring them nearer, and repulsion when it tends to separate them.

In all cases, however, the action and reaction between the bodies are equal and opposite.

THE THIRD LAW TRUE OF ACTION AT A DISTANCE.—The fact that a magnet draws iron towards it was noticed by the ancients, but no attention was paid to the force with which the iron attracts the magnet. Newton, however, by placing the magnet in one vessel and the iron in another, and floating both vessels in water so as to touch each other, showed experimentally that as neither vessel was able to propel the other along with itself through the water, the attraction of the

iron on the magnet must be equal and opposite to that of the magnet on the iron, both being equal to the pressure between the two vessels.

Having given this experimental illustration Newton goes on to point out the consequence of denying the truth of this law. For instance, if the attraction of any part of the earth, say a mountain, upon the remainder of the earth were greater or less than that of the remainder of the earth upon the mountain, there would be a residual force, acting upon the system of the earth and the mountain as a whole, which would cause it to move off, with an ever-increasing velocity, through infinite space.

NEWTON'S PROOF NOT EXPERIMENTAL. —This is contrary to the first law of motion, which asserts that a body does not change its state of motion unless acted on by *external* force. It cannot be affirmed to be contrary to experience, for the effect of an inequality between the attraction of the earth on the mountain

and the mountain on the earth would be the same as that of a force equal to the difference of these attractions acting in the direction of the line joining the center of the earth with the mountain.

If the mountain were at the equator the earth would be made to rotate about an axis parallel to the axis about which it would otherwise rotate, but not passing exactly through the center of the earth's mass.

If the mountain were at one of the poles, the constant force parallel to the earth's axis would cause the orbit of the earth about the sun to be slightly shifted to the north or south of a plane passing through the center of the sun's mass.

If the mountain were at any other part of the earth's surface its effect would be partly of the one kind and partly of the other.

Neither of these effects, unless they were very large, could be detected by direct astronomical observations, and the indirect method of detecting small

forces, by their effect in slowly altering the elements of a planet's orbit, presupposes that the law of gravitation is known to be true. To prove the laws of motion by the law of gravitation would be an inversion of scientific order. We might as well prove the law of addition of numbers by the differential calculus.

We cannot, therefore, regard Newton's statement as an appeal to experience and observation, but rather as a deduction of the third law of motion from the first.

CHAPTER IV.

ON THE PROPERTIES OF THE CENTER OF MASS OF A MATERIAL SYSTEM.

Definition of a Mass-Vector.—We have seen that a vector represents the operation of carrying a tracing point from a given origin to a given point.

Let us define a mass-vector as the operation of carrying a given mass from

the origin to the given point. The direction of the mass-vector is the same as that of the vector of the mass, but its magnitude is the product of the mass into the vector of the mass.

Thus if $\overline{OA}$ is the vector of the mass A, the mass-vector is $\overline{OA}.A$.

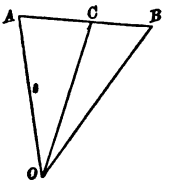
CENTER OF MASS OF TWO PARTICLES. —If A and B are two masses, and if a point C be taken in the straight line $\overline{AB}$, so that $\overline{BC}$ is to $\overline{CA}$ as A to B, then the mass-vector of a mass $A+B$ placed at C is equal to the sum of the mass-vectors of A and B.

For $\overline{OA}.A+\overline{OB}.B=(\overline{OC}+\overline{CA})A+(\overline{OC}+\overline{CB})B.$

$$=\overline{OC}(A+B)+\overline{CA}.A+\overline{CB}.B.$$

Now the mass-vectors $\overline{CA}.A$ and $\overline{CB}.B$ are equal and opposite, and so destroy each other, so that $\overline{OA}.A+\overline{OB}.B=\overline{OC}(A+B)$

FIG. 7.



or, C is a point such that if the masses of A and B were concentrated at C, their mass-vector from any origin O would be the same as when A and B are in their actual positions. The point C is called the *Center of Mass* of A and B.

CENTER OF MASS OF A SYSTEM.—If the system consists of any number of particles, we may begin by finding the center of mass of any two particles, and substituting for the two particles a particle equal to their sum placed at their center of mass. We may then find the center of mass of this particle, together with the third particle of the system, and place the sum of the three particles at this point, and so on till we

have found the center of mass of the whole system.

The mass-vector drawn from any origin to a mass, equal to that of the whole system placed at the center of mass of the system, is equal to the sum of the mass-vectors drawn from the same origin to all the particles of the system.

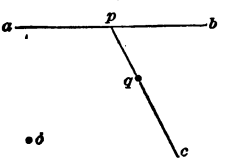
It follows, from the proof in "Center of Mass of Two Particles," that the point found by the construction here given satisfies this condition. It is plain from the condition itself that only one point can satisfy it. Hence the construction must lead to the same result, as to the position of the center of mass, in whatever order we take the particles of the system.

The center of mass is therefore a definite point in the diagram of the configuration of the system. By assigning to the different points in the diagrams of displacement, velocity, total acceleration, and rate of acceleration, the masses of the bodies to which they correspond, we

may find in each of these diagrams a point which corresponds to the center of mass, and indicates the displacement, velocity, total acceleration of the center of mass.

MOMENTUM REPRESENTED AS THE RATE OF CHANGE OF A MASS-VECTOR.— In the diagram of velocities, if the points o, a, b, c, correspond to the velocities of the origin O and the bodies A, B, C, and if p be the center of mass of A and B placed at a and b respectively, and if q is the center of mass of A+B placed at p and C at c, then q will be the center of mass of the system of bodies A, B, C, at a, b, c, respectively.

FIG. 8.



The velocity of A with respect to O is indicated by the vector $o\,a$, and that of

B and C by $\overline{o\,b}$ and $\overline{o\,c}$. $\overline{o\,p}$ is the velocity of the center of mass of A and B, and $\overline{o\,q}$ that of the center of mass of A, B, and C, with respect to O.

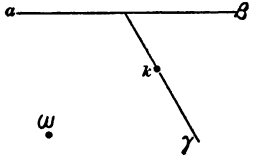
The momentum of A with respect to O is the product of the velocity into the mass, or $\overline{o\,a}$.A, or what we have already called the mass-vector, drawn from *o* to the mass A at *a*. Similarly the momentum of any other body is the mass-vector drawn from *o* to the point on the diagram of velocities corresponding to that body, and the momentum of the mass of the system concentrated at the center of mass is the mass-vector drawn from *o* to the whole mass at *q*.

Since, therefore, a mass-vector in the diagram of velocities is what we have already defined as a momentum, we may state the property proved in "Center of Mass of a System," in terms of momenta, thus: The momentum of a mass equal to that of the whole system, moving with the velocity of the center of mass of the system, is equal in magnitude and parallel in direction to the sum of the

momenta of all the particles of the system.

Effect of External Forces on the Motion of the Center of Mass.—In the same way in the diagram of Total Acceleration the vectors ωa, $\omega \beta$, drawn from the origin, represent the change of velocity of the bodies A, B, &c., during a certain interval of time. The corresponding mass-vectors, ωa.A, $\omega \beta$.B., &c., represent the corresponding changes

Fig. 9.



of momentum, or, by the second law of motion, the impulses of the forces acting on these bodies during that interval of time. If k is the center of mass of the system ωk is the change of velocity during the interval, and ωk (A+B+C) is the momentum generated in the mass

concentrated at the center of gravity. Hence, by "Center of Mass of a System," the change of momentum of the imaginary mass equal to that of the whole system concentrated at the center of mass is equal to the sum of the changes of momentum of all the different bodies of the system.

In virtue of the second law of motion we may put this result in the following form :

The effect of the forces acting on the different bodies of the system in altering the motion of the center of mass of the system is the same as if all these forces had been applied to a mass equal to the whole mass of system, and coinciding with its center of mass.

THE MOTION OF THE CENTER OF MASS OF A SYSTEM IS NOT AFFECTED BY THE MUTUAL ACTION OF THE PARTS OF THE SYSTEM.--For if there is an action between two parts of the system, say A and B, the action of A on B is always, by the third law of motion, equal and

opposite to the reaction of B on A. The momentum generated in B by the action of A during any interval is therefore equal and opposite to that generated in A by the reaction of B during the same interval, and the motion of the center of mass of A and B is therefore not affected by their mutual action.

We may apply the result of the last article to this case and say, that since the forces on A and on B arising from their mutual action are equal and opposite, and since the effect of these forces on the motion of the center of mass of the system is the same as if they had been applied to a particle whose mass is equal to the whole mass of the system, and since the effect of two forces equal and opposite to each other is zero, the motion of the center of mass will not be affected.

First and Second Laws of Motion. —This is a very important result. It enables us to render more precise the enunciation of the first and second laws

of motion, by defining that by the velocity of a body is meant the velocity of its center of mass. The body may be rotating, or it may consist of parts, and be capable of changes of configuration, so that the motions of different parts may be different, but we can still assert the laws of motion in the following form :

Law I. The center of mass of the system perseveres in its state of rest, or of uniform motion in a straight line, except in so far as it is made to change that state by forces acting on the system from without.

Law II. The change of momentum of the system during any interval of time is measured by the sum of the impulses of the external forces during that interval.

Method of Treating Systems of Molecules.—When the sytem is made up of parts which are so small that we cannot observe them, and whose motions are so rapid and so variable that even if we

could observe them we could not describe them, we are still able to deal with the motion of the center of mass of the system, because the internal forces which cause the variation of the motion of the parts do not affect the motion of the center of mass.

By the Introduction of the Idea of Mass we pass from Point-Vectors, Point Displacements, Velocities, Total Accelerations, and Rates of Acceleration, to Mass-Vectors, Mass Displacements, Momenta, Impulses, and Moving Forces.—In the diagram of rates of acceleration (Fig. 9, "Effect of External Forces on the Motion of the Center of Mass)," the vectors $\overline{\omega\alpha}$, $\overline{\omega\beta}$, &c., drawn from the origin, represent the rates of acceleration of the bodies A, B, &c., at a given instant, with respect to that of the origin O.

The corresponding mass-vectors, $\overline{\omega\alpha}.A$, $\overline{\omega\beta}.B$, &c., represent the forces acting on the bodies A, B, &c.

We sometimes speak of several forces

pears to be the direction of the motion of the hands of a watch.

If the area is in one plane, the path of the particle must be divided into portions so small that each coincides sensibly with a straight line, and the mass-areas corresponding to these portions must be added together by the rule for the addition of vectors.

ANGULAR MOMENTUM.—The rate of change of a mass-area is twice the mass of the particle into the triangle, whose vertex is the origin and whose base is the velocity of the particle measured along the line through the particle in the direction of its motion. The direction of this mass-area is indicated by the normal drawn according to the rule given above.

The rate of change of the mass-area of a particle is called the Angular Momentum of the particle about the origin, and the sum of the angular momenta of all the particles is called the angular momentum of the system about the origin.

The angular momentum of a material system with respect to a point is, therefore, a quantity having a definite direction as well as a definite magnitude.

The definition of the angular momentum of a particle about a point may be expressed somewhat differently—as the product of the momentum of the particle with respect to that point into the perpendicular from that point on the line of motion of the particle at that instant.

MOMENT OF A FORCE ABOUT A POINT. —The rate of increase of the angular momentum of a particle is the continued product of the rate of acceleration of the velocity of the particle into the mass of the particle into the perpendicular from the origin on the line through the particle along which the acceleration takes place. In other words, it is the product of the moving force acting on the particle into the perpendicular from the origin on the line of action of this force.

Now the produce of a force into the perpendicular from the origin on its line

of action is called the Moment of the Force about the origin. The axis of the moment, which indicates its direction, is a vector drawn perpendicular to the plane passing through the force and the origin, and in such a direction that looking along this line in the direction in which it is drawn, the force tends to move the particle round the origin in the direction of the hands of a watch.

Hence the rate of change of the angular momentum of a particle about the origin is measured by the moment of the force which acts on the particle about that point.

The rate of change of the angular momentum of a material system about the origin is in like manner measured by the geometric sum of the moments of the forces which act on the particles of the system.

Conservation of Angular Momentum.—Now consider any two particles of the system. The forces acting on these two particles, arising from their

mutual action, are equal, opposite, and in the same straight line. Hence the moments of these forces about any point as origin are equal, opposite, and about the same axis. The sum of these moments is therefore zero. In like manner the mutual action between every other pair of particles in the system consists of two forces, the sum of whose moments is zero.

Hence the mutual action between the bodies of a material system does not affect the geometric sum of the moments of the forces. The only forces, therefore, which need be considered in finding the geometric sum of the moments are those which are external to the system—that is to say, between the whole or any part of the system and bodies not included in the system.

The rate of change of the angular momentum of the system is therefore measured by the geometric sum of the moments of the external forces acting on the system.

If the directions of all the external

forces pass through the origin, their moments are zero, and the angular momentum of the system will remain constant.

When a planet describes an orbit about the sun, the direction of the mutual action between the two bodies always passes through their common center of mass. Hence the angular momentum of either body about their common center of mass remains constant, so far as these two bodies only are concerned, though it may be affected by the action of other planets. If, however, we include all the planets in the system, the geometric sum of their angular momenta about their common center of mass will remain absolutely constant, whatever may be their mutual actions, provided no force arising from bodies external to the whole solar system acts in an unequal manner upon the different members of the system.

CHAPTER V.

ON WORK AND ENERGY.

DEFINITIONS.—Work *is the act of producing a change of configuration in a system in opposition to a force which resists that change.*

Energy *is the capacity of doing work.*

When the nature of a material system is such that if, after the system has undergone any series of changes, it is brought back in any manner to its original state, the whole work done by external agents on the system is equal to the whole work done by the system in overcoming external forces, the system is called a CONSERVATIVE SYSTEM.

PRINCIPLE OF CONSERVATION OF ENERGY.—The progress of physical science has led to the discovery and investigation of different forms of energy, and to the establishment of the doctrine that all material systems may be regarded as conservative systems, provided that all

the different forms of energy which exist in these systems are taken into account.

This doctrine, considered as a deduction from observation and experiment can, of course, assert no more than that no instance of a non-conservative system has hitherto been discovered.

As a scientific or science-producing doctrine, however, it is always acquiring additional credibility from the constantly increasing number of deductions which have been drawn from it, and which are found in all cases to be verified by experiment.

In fact the doctrine of the Conservation of Energy is the one generalized statement which is found to be consistent with fact, not in one physical science only, but in all.

When once apprehended it furnishes to the physical inquirer a principle on which he may hang every known law relating to physical actions, and by which he may be put in the way to discover the relations of such actions in new branches of science.

For such reasons the doctrine is commonly called the Principle of the Conservation of Energy.

General Statement of the Principle of the Conservation of Energy. —*The total energy of any material system is a quantity which can neither be increased nor diminished by any action between the parts of the system, though it may be transformed into any of the forms of which energy is susceptible.*

If, by the action of some agent external to the system, the configuration of the system is changed, while the forces of the system resist this change of configuration, the external agent is said to do work on the system. In this case the energy of the system is increased by the amount of work done on it by the external agent.

If, on the contrary, the forces of the system produce a change of configuration which is resisted by the external agent, the system is said to do work on the external agent, and the energy of the

system is diminished by the amount of work which it does.

Work, therefore, is a transference of energy from one system to another; the system which gives out energy is said to do work on the system which receives it, and the amount of energy given out by the first system is always exactly equal to that received by the second.

If, therefore, we include both systems in one larger system, the energy of the total system is neither increased nor diminished by the action of the one partial system on the other.

MEASUREMENT OF WORK.—Work done by an external agent on a material system may be described as a change in the configuration of the system taking place under the action of an external force tending to produce that change.

Thus, if one pound is lifted one foot from the ground by a man in opposition to the force of gravity, a certain amount of work is done by the man, and this quantity is known among engineers as one foot-pound.

Here the man is the external agent, the material system consists of the earth and the pound, the change of configuration is the increase of the distance between the matter of the earth and the matter of the pound, and the force is the upward force exerted by the man in lifting the pound, which is equal and opposite to the weight of the pound. To raise the pound a foot higher would, if gravity were a uniform force, require exactly the same amount of work. It is true that gravity is not really uniform, but diminishes as we ascend from the earth's surface, so that a foot-pound is not an accurately know quantity, unless we specify the intensity of gravity at the place. But for the purpose of illustration we may assume that gravity is uniform for a few feet of ascent, and in that case the work done in lifting a pound would be one foot-pound for every foot the pound is lifted.

To raise twenty pounds of water ten feet high requires 200 foot-pounds of work. To raise one pound ten feet high

requires ten foot-pounds, and as there are twenty pounds the whole work is twenty times as much, or two hundred foot-pounds.

The quantity of work done is, therefore, proportional to the product of the numbers representing the force exerted and the displacement in the direction of the force.

In the case of a foot-pound the force is the weight of a pound—a quantity which, as we know, is different in different places. The weight of a pound expressed in absolute measure is numerically equal to the intensity of gravity, the quantity denoted by g, the value of which in poundals to the pound varies from 32.227 at the pole to 32.117 at the equator, and diminishes without limit as we recede from the earth. In dynes to the gramme it varies from 978.1 to 983.1. Hence, in order to express work in a uniform and consistent manner, we must multiply the number of foot-pounds by the number representing the intensity of gravity at the place. The work is thus

reduced to foot-poundals. We shall always understand work to be measured in this manner and reckoned in foot-poundals when no other system of measurement is mentioned. When work is expressed in foot-pounds the system is that of *gravitation-measures*, which is not a complete system unless we also know the intensity of gravity at the place.

In the metrical system the unit of work is the Erg, which is the work done by a dyne acting through a centimeter. There are 421393.8 ergs in a foot-poundal.

Potential Energy.—The work done by a man in raising a heavy body is done in overcoming the attraction between the earth and that body. The energy of the material system, consisting of the earth and the heavy body, is thereby increased. If the heavy body is the leaden weight of a clock, the energy of the clock is increased by winding it up, so that the clock is able to go for a week in spite of the friction of the wheels and the resist-

ance of the air to the motion of the pendulum, and also to give out energy in other forms, such as the communication of the vibrations to the air, by which we hear the ticking of the clock.

When a man winds up a watch he does work in changing the form of the mainspring by coiling it up. The energy of the mainspring is thereby increased, so that as it uncoils itself it is able to keep the watch going.

In both these cases the energy communicated to the system depends upon a change of configuration.

KINETIC ENERGY.—But in a very important class of phenomena the work is done in changing the velocity of the body on which it acts. Let us take as a simple case that of a body moving without rotation under the action of a force. Let the mass of the body be M pounds, and let a force of F poundals act on it in the line of motion during an interval of time, T seconds. Let the velocity at the beginning of the interval be V and

that at the end V' feet per second, and let the distance traveled by the body during the time be S feet. The original momentum is MV, and the final momentum is MV', so that the increase of momentum is $M(V'-V)$, and this, by the second law of motion is equal to FT, the *impulse* of the force F acting for the time T. Hence

$$FT=M(V'-V). \qquad (1)$$

Since the velocity increases uniformly with the time, the mean velocity is the arithmetical mean of the original and final velocities, or $\frac{1}{2}(V'+V)$.

We can also determine the mean velocity by dividing the space S by the time T, during which it is described.

$$\text{Hence } \frac{S}{T}=\frac{1}{2}(V'+V). \qquad (2)$$

Multiplying the corresponding members of equations (1) and (2) each by each we obtain—

$$FS=\tfrac{1}{2}MV'^2-\tfrac{1}{2}MV^2 \qquad (3)$$

Here FS is the work done by the force F acting on the body while it moves

through the space S in the direction of the force, and this is equal to the excess of $\frac{1}{2}$ MV'^{2} above $\frac{1}{2}$ MV^2. If we call $\frac{1}{2}$ MV^2, or half the product of the mass, into the square of the velocity, the *kinetic energy* of the body at first, then $\frac{1}{2}$ MV'^{2} will be the kinetic energy after the action of the force F through the space S. The energy is here expressed in foot-poundals.

We may now express the equation in words by saying that the work done by the force F in changing the motion of the body is measured by the increase of the kinetic energy of the body during the time that the force acts.

We have proved that this is true when the interval of time is so small that we may consider the force as constant during that time, and the mean velocity during the interval as the arithmetical mean of the velocities at the beginning and end of the interval. This assumption, which is exactly true when the force is constant, however long the interval may be, becomes in every case more and more

nearly true as the interval of time taken becomes smaller and smaller. By dividing the whole time of action into small parts, and proving that in each of these the work done is equal to the increase of the kinetic energy of the body, we may, by adding the successive portions of the work and the successive increments of energy, arrive at the result that the total work done by the force is equal to the total increase of kinetic energy.

If the force acts on the body in the direction opposite to its motion, the kinetic energy of the body will be diminished instead of being increased, and the force, instead of doing work on the body, will act as a resistance, which the body, in its motion, overcomes. Hence a moving body, as long as it is in motion, can do work in overcoming resistance, and the work done by the moving body is equal to the diminution of its kinetic energy, till at last, when the body is brought to rest, its kinetic energy is exhausted, and the whole work it has done is then equal

to the whole kinetic energy which it had at first.

We now see the appropriateness of the name *kinetic energy*, which we have hitherto used merely as a name to denote the product $\frac{1}{2}MV^2$. For the energy of a body has been defined as the capacity which it has of doing work, and it is measured by the work which it can do. The *kinetic* energy of a body is the energy it has in virtue of being in *motion*, and we have now shown that its value is expressed by $\frac{1}{2}MV^2$ or $\frac{1}{2}MV \times V$, that is, half the product of its momentum into its velocity.

Oblique Forces.—If the force acts on the body at right angles to the direction of its motion it does not work on the body, and it alters the direction but not the magnitude of the velocity. The kinetic energy, therefore, which depends on the square of the velocity, remains unchanged.

If the direction of the force is neither coincident with, nor at right angles to,

that of the motion of the body we may resolve the force into two components, one of which is at right angles to the direction of motion, while the other is in the direction of motion (or in the opposite direction).

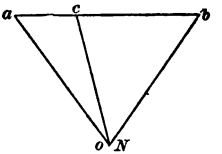
The first of these components may be left out of consideration in all calculations about energy, since it neither does work on the body nor alters its kinetic energy.

The second component is that which we have already considered. When it is in the direction of motion it increases the kinetic energy of the body by the amount of work which it does on the body. When it is in the opposite direction the kinetic energy of the body is diminished by the amount of work which the body does against the force.

Hence in all cases the increase of kinetic energy is equal to the work done on the body by external agency, and the diminution of kinetic energy is equal to the work done by the body against external resistance.

KINETIC ENERGY OF TWO PARTICLES REFERRED TO THEIR CENTER OF MASS. —The kinetic energy of a material system is equal to the kinetic energy of a mass equal to that of the system moving with the velocity of the center of mass of the system, together with the kinetic energy due to the motion of the parts of the system relative to its center of mass.

FIG. 10.



Let us begin with the case of two particles whose masses are A and B, and whose velocities are represented in the diagram of velocities by the lines *o a* and *o b*. If *c* is the center of mass of a particle equal to A placed at *a*, and a particle equal to B placed at *b*, then *o c* will represent the velocity of the center of mass of the two particles.

The kinetic energy of the system is the sum of the kinetic energies of the particles, or

$$T=\tfrac{1}{2}\,A\,\overline{oa}^2+\tfrac{1}{2}\,B\,\overline{ob}^2.$$

Expressing $\overline{oa}^2$ and $\overline{ob}^2$ in terms of $o\,c$, $c\,a$ and $c\,b$ and the angle $o\,c\,a=\theta$.

$$T=\tfrac{1}{2}\,A\,\overline{oc}^2+\tfrac{1}{2}\,A\,\overline{ca}^2-A\,o\,c.c\,a\cos\theta.$$
$$+\tfrac{1}{2}\,B\,\overline{oc}^2+\tfrac{1}{2}\,B\,\overline{cb}^2-B\,o\,c.c\,b\,cos\,\theta.$$

But since c is the center of mass of A at a, and B at b,

$$A.\,\overline{ca}+B\,\overline{cb}=0.$$

Hence adding

$$T=\tfrac{1}{2}\,(A+B)\,\overline{oc}^2+\tfrac{1}{2}\,A\,\overline{ca}^2+\tfrac{1}{2}\,B\,\overline{cb}^2,$$

or, the kinetic energy of the system of two particles A and B is equal to that of a mass equal to (A + B) moving with the velocity of the center of mass, together with that of the motion of the particles relative to the center of mass.

Kinetic Energy of a Material System Referred to its Center of Mass. —We have begun with the case of two

particles, because the motion of a particle is assumed to be that of its center of mass, and we have proved our proposition true for a system of two particles. But if the proposition is true for each of two material systems taken separately, it must be true of the system which they form together. For if we now suppose oa and ob to represent the velocities of the centers of mass of two material systems A and B, then $o\,c$ will represent the velocity of the center of mass of the combined system A+B, and if T_A represents the kinetic energy of the motion of the system A relative to its own center of mass, and T_B the same for the system B, then if the proposition is true for the systems A and B taken separately, the kinetic energy of A is

$$\tfrac{1}{2}\,A\,\overline{oa}^2+T_A,$$

and that of B

$$\tfrac{1}{2}\,B\,\overline{ob}^2+T_B.$$

The kinetic energy of the whole is, therefore,

$$\tfrac{1}{2}\,A\,\overline{oa}^2+\tfrac{1}{2}\,B\,\overline{ob}^2+T_A+T_B,$$

or,

$$\tfrac{1}{2}(A+B)\overline{oc}^2+\tfrac{1}{2}A\overline{ca}^2+T_A+\tfrac{1}{2}B\overline{cb}^2+T_B.$$

The first term represents the kinetic energy of a mass equal to that of the whole system moving with the velocity of the center of mass of the whole system.

The second and third terms, taken together, represent the kinetic energy of the system A relative to the center of gravity of the whole system, and the fourth and fifth terms represent the same for the system B.

Hence if the proposition is true for the two systems A and B taken separately, it is true for the system compounded of A and B. But we have proved it true for the case of two particles; it is, therefore, true for three, four, or any other number of particles, and therefore for any material system.

The kinetic energy of a system referred to its center of mass is less than its kinetic energy when referred to any other point.

For the latter quantity exceeds the

former by a quantity equal to the kinetic energy of a mass equal to that of the whole system moving with the velocity of the center of mass relative to the other point, and since all kinetic energy is essentially positive, this excess must be positive.

AVAILABLE KINETIC ENERGY.—We have already seen in "The Motion of the Center of Mass," etc., that the mutual action between the parts of a material system cannot change the velocity of the center of mass of the system. Hence that part of the kinetic energy of the system which depends on the motion of the center of mass cannot be affected by any action internal to the system. It is therefore, impossible, by means of the mutual action of the parts of the system, to convert this part of the energy into work. As far as the system itself is concerned, this energy is unavailable. It can be converted into work only by means of the action between this system and some other material system external to it.

Hence if we consider a material system unconnected with any other system, its available kinetic energy is that which is due to the motions of the parts of the system relative to its center of mass.

Let us suppose that the action between the parts of the system is such that after a certain time the configuration of the system becomes invariable, and let us call this process the solidification of the system. We have shown that the angular momentum of the whole system is not changed by any mutual action of its parts. Hence if the original angular momentum is zero, the system, when its form becomes invariable, will not rotate about its center of mass, but if it moves at all will move parallel to itself, and the parts will be at rest relative to the center of mass. In this case, therefore, the whole available energy will be converted into work by the mutual action of the parts during the solidification of the system.

If the system has angular momentum, it will have the same angular momentum

when solidified. It will therefore rotate about its center of mass, and will therefore still have energy of motion relative to its center of mass, and this remaining kinetic energy has not been converted into work.

But if the parts of the system are allowed to separate from one another in directions perpendicular to the axis of the angular momentum of the system, and if the system when thus expanded is solidified, the remaining kinetic energy of rotation round the center of mass will be less and less the greater the expansion of the system, so that by sufficiently expanding the system we may make the remaining kinetic energy as small as we please, so that the whole kinetic energy relative to the center of mass of the system may be converted into work within the system.

POTENTIAL ENERGY.—The potential energy of a material system is the capacity which it has of doing work depending on other circumstances than the

motion of the system. In other words, potential energy is that energy which is not kinetic.

In the theoretical material system which we build up in our imagination from the fundamental ideas of matter and motion, there are no other conditions present except the configuration and motion of the different masses of which the system is composed. Hence in such a system the circumstances upon which the energy must depend are motion and configuration only, so that, as the kinetic energy depends on the motion, the potential energy must depend on the configuration.

In many real material systems we know that part of the energy does depend on the configuration. Thus the mainspring of a watch has more energy when coiled up than when partially uncoiled, and two bar magnets have more energy when placed side by side with their similar poles turned the same way than when their dissimilar poles are placed next each other.

ELASTICITY.—In the case of the spring we may trace the connection between the coiling of the spring and the force which it exerts somewhat further by conceiving the spring divided (in imagination) into very small parts or elements. When the spring is coiled up, the form of each of these small parts is altered, and such an alteration of the form of a solid body is called a Strain.

In solid bodies strain is accompanied with internal force or stress; those bodies in which the stress depends simply on the strain are called Elastic, and the property of exerting stress when strained is called Elasticity.

We thus find that the coiling of the spring involves the strain of its elements, and that the external force which the spring exerts is the resultant of the stresses in its elements.

We thus substitute for the immediate relation between the coiling of the spring and the force which it exerts a relation between the strains and stresses of the elements of the spring; that is to say, for

a single displacement and a single force, the relation between which may in some cases be of an exceedingly complicated nature, we substitute a multitude of strains and an equal number of stresses, each strain being connected with its corresponding stress by a much more simple relation.

But when all is done, the nature of the connection between configuration and force remains as mysterious as ever. We can only admit the fact, and if we call all such phenomena phenomena of elasticity, we may find it very convenient to classify them in this way, provided we remember that by the use of the word elasticity we do not profess to explain the cause of the connection between configuration and energy.

Action at a Distance.—In the case of the two magnets there is no visible substance connecting the bodies between which the stress exists. The space between the magnets may be filled with air or with water, or we may place the

magnets in a vessel and remove the air by an air-pump, till the magnets are left in what is commonly called a vacuum, and yet the mutual action of the magnets will not be altered. We may even place a solid plate of glass or metal or wood between the magnets, and still we find that their mutual action depends simply on their relative position, and is not perceptibly modified by placing any substance between them, unless that substance is one of the magnetic metals. Hence the action between the magnets is commonly spoken of as *action at a distance.*

Attempts have been made, with a certain amount of success,* to analyze this action at a distance into a continuous distribution of stress in an invisible medium, and thus to establish an analogy between the magnetic action and the action of a spring or a rope in transmitting force; but still the general fact that strains or changes of configuration are

* See Clerk Maxwell's "Treatise on Electricity and Magnetism," Vol. II., Art. 641.

accompanied by stresses or internal forces, and that thereby energy is stored up in the system so strained, remains an ultimate fact which has not yet been explained as the result of any more fundamental principle.

THEORY OF POTENTIAL ENERGY MORE COMPLICATED THAN THAT OF KINETIC ENERGY.—Admitting that the energy of a material system may depend on its configuration, the mode in which it so depends may be much more complicated than the mode in which the kinetic energy depends on the motion of the system. For the kinetic energy may be calculated from the motion of the parts of the system by an invariable method. We multiply the mass of each part by half the square of its velocity, and take the sum of all such products. But the potential energy arising from the mutual action of two parts of the system may depend on the relative position of the parts in a manner which may be different in different instances. Thus when two

billiard balls approach each other from a distance, there is no sensible action between them till they come so near one another that certain parts appear to be in contact. To bring the centers of the two balls nearer, the parts in contact must be made to yield, and this requires the expenditure of work.

Hence in this case the potential energy is constant for all distances greater than the distance of first contact, and then rapidly increases when the distance is diminished.

The force between magnets varies with the distance in a very different manner, and in fact we find that it is only by experiment that we can ascertain the form of the relation between the configuration of a system and its potential energy.

APPLICATION OF THE METHOD OF ENERGY TO THE CALCULATION OF FORCES. —A complete knowledge of the mode in which the energy of a material system varies when the configuration and motion

of the system are made to vary is mathematically equivalent to a knowledge of all the dynamical properties of the system. The mathematical methods by which all the forces and stresses in a moving system are deduced from the single mathematical formula which expresses the energy as a function of the variables have been developed by Lagrange, Hamilton, and other eminent mathematicians, but it would be difficult even to describe them in terms of the elementary ideas to which we restrict ourselves in this book. An outline of these methods is given in my treatise on Electricity, Part IV., Chapter V., Article 553, and the application of these dynamical methods to electro-magnetic phenomena is given in the chapters immediately following.

But if we consider only the case of a system at rest it is easy to see how we can ascertain the forces of the system when we know how its energy depends on its configuration.

For let us suppose that an agent ex-

ternal to the system produces a displacement from one configuration to another, then if in the new configuration the system possess more energy than it did at first, it can have received this increase of energy only from the external agent. This agent must therefore have done an amount of work equal to the increase of energy. It must therefore have exerted force in the direction of the displacement, and the mean value of this force, multiplied into the displacement, must be equal to the work done. Hence the mean value of the force may be found by dividing the increase of energy by the displacement.

If the displacement is large this force may vary considerably during the displacement, so that it may be difficult to calculate its mean value; but since the force depends on the configuration, if we make the displacement smaller and smaller the variation of the force will become smaller and smaller, so that at last the force may be regarded as sensibly constant during the displacement.

If, therefore, we calculate for a given configuration the *rate* at which the energy increases with the displacement, by a method similar to that described "On the Measurement of Velocity when Variable;" "Diagram of Velocities," and "On the Rate of Acceleration," this rate will be numerically equal to the force exerted by the external agent in the direction of the displacement.

If the energy diminishes instead of increasing as the displacement increases, the system must do work on the external agent, and the force exerted by the external agent must be in the direction opposite to that of displacement.

SPECIFICATION OF THE DIRECTION OF FORCES.—In treatises on dynamics the forces spoken of are usually those exerted by the external agent on the material system. In treatises on electricity, on the other hand, the forces spoken of are usually those exerted by the electrified system against an external agent which prevents the system from moving. It is

necessary, therefore, in reading any statement about forces, to ascertain whether the force spoken of is to be regarded from the one point of view or the other.

We may in general avoid any ambiguity by viewing the phenomenon as a whole, and speaking of it as a stress exerted between two points or bodies, and distinguishing it as a tension or a pressure, an attraction or a repulsion, according to its direction, "Action and Reaction are the Partial Aspects of a Stress."

Application to a System in Motion. —It thus appears that from a knowledge of the potential energy of a system in every possible configuration we may deduce all the external forces which are required to keep the system in that configuration. If the system is at rest, and if these external forces are the actual forces, the system will remain in equilibrium. If the system is in motion the force acting on each particle is that arising from the connections of the sys-

tem (equal and opposite to the external force just calculated), together with any external force which may be applied to it. Hence a complete knowledge of the mode in which the potential energy varies with the configuration would enable us to predict every possible motion of the system under the action of given external forces, provided we were able to overcome the purely mathematical difficulties of the calculation.

Application of the Method of Energy to the Investigation of Real Bodies.—When we pass from abstract dynamics to physics—from material systems, whose only properties are those expressed by their definitions, to real bodies, whose properties we have to investigate—we find that there are many phenomena which we are not able to explain as changes in the configuration and motion of a material system.

Of course if we begin by assuming that the real bodies are systems composed of matter which agrees in all re-

spects with the definitions we have laid down, we may go on to assert that all phenomena are changes of configuration and motion, though we are not prepared to define the kind of configuration and motion by which the particular phenomena are to be explained. But in accurate science such asserted explanations must be estimated, not by their promises, but by their performances. The configuration and motion of a system are facts capable of being described in an accurate manner, and therefore, in order that the explanation of a phenomenon by the configuration and motion of a material system may be admitted as an addition to our scientific knowledge, the configurations, motions, and forces must be specified, and shown to be consistent with known facts, as well as capable of accounting for the phenomenon.

VARIABLES ON WHICH THE ENERGY DEPENDS.—But even when the phenomena we are studying have not yet been explained dynamically, we are still

able to make great use of the principle of the conservation of energy as a guide to our researches.

To apply this principle, we in the first place assume that the quantity of energy in a material system depends on the state of that system, so that for a given state there is a definite amount of energy.

Hence the first step is to define the different states of the system, and when we have to deal with real bodies we must define their state with respect not only to the configuration and motion of their visible parts, but if we have reason to suspect that the configuration and motion of their invisible particles influence the visible phenomenon, we must devise some method of estimating the energy thence arising.

Thus pressure, temperature, electric potential, and chemical composition are variable quantities, the values of which serve to specify the state of a body, and in general the energy of the body depends on the values of these and other variables.

ENERGY IN TERMS OF THE VARIABLES. —The next step in our investigation is to determine how much work must be done by external agency on the body in order to make it pass from one specified state to another.

For this purpose it is sufficient to know the work required to make the body pass from a particular state, which we may call the *standard state*, into any other specified state. The energy in the latter state is equal to that in the standard state, together with the work required to bring it from the standard state into the specified state. The fact that this work is the same through whatever series of states the system has passed from the standard state to the specified state is the foundation of the whole theory of energy.

Since all the phenomena depend on the variations of the energy of the body, and not on its total value, it is unnecessary, even if it were possible, to form any estimate of the energy of the body in its standard state.

THEORY OF HEAT.—One of the most important applications of the principle of the conservation of energy is to the investigation of the nature of heat.

At one time it was supposed that the difference between the states of a body when hot and when cold was due to the presence of a substance called caloric, which existed in greater abundance in the body when hot than when cold. But the experiments of Rumford on the heat produced by the friction of metal, and of Davy on the melting of ice by friction, have shown that when work is spent in overcoming friction, the amount of heat produced is proportional to the work spent.

The experiments of Hirn have also shown that when heat is made to do work in a steam-engine, part of the heat disappears, and that the heat which disappears is proportional to the work done.

A very careful measurement of the work spent in friction, and of the heat produced, has been made by Joule, who finds that the heat required to raise one

pound of water from 39° F. to 40° F. is equivalent to 772 foot-pounds of work at Manchester, or 24,858 foot-poundals.

From this we may find that the heat required to raise one gramme of water from 3° C. to 4° C. is 42,000,000 ergs.

HEAT A FORM OF ENERGY.—Now, since heat can be produced it cannot be a substance; and since whenever mechanical energy is lost by friction there is a production of heat, and whenever there is a gain of mechanical energy in an engine there is a loss of heat; and since the quantity of energy lost or gained is proportional to the quantity of heat gained or lost, we conclude that heat is a form of energy.

We have also reasons for believing that the minute particles of a hot body are in a state of rapid agitation, that is to say, that each particle is always moving very swiftly, but that the direction of its motion alters so often that it makes little or no progress from one region to another.

If this be the case, a part, and it may be a very large part, of the energy of a hot body must be in the form of kinetic energy.

But for our present purpose it is unnecessary to ascertain in what form energy exists in a hot body; the most important fact is that energy may be measured in the form of heat, and since every kind of energy may be converted into heat, this gives us one of the most convenient methods of measuring it.

Energy Measured as Heat.—Thus when certain substances are placed in contact chemical actions take place, the substances combine in a new way, and the new group of substances has different chemical properties from the original group of substances. During this process mechanical work may be done by the expansion of the mixture, as when gunpowder is fired; an electric current may be produced, as in the voltaic battery; and heat may be generated, as in most chemical actions.

The energy given out in the form of mechanical work may be measured directly, or it may be transformed into heat by friction. The energy spent in producing the electric current may be estimated as heat by causing the current to flow through a conductor of such a form that the heat generated in it can easily be measured. Care must be taken that no energy is transmitted to a distance in the form of sound or radiant heat without being duly accounted for.

The energy remaining in the mixture, together with the energy which has escaped, must be equal to the original energy.

Andrews, Favre and Silbermann, and others, have measured the quantity of heat produced when a certain quantity of oxygen or of chlorine combines with its equivalent of other substances. These measurements enable us to calculate the excess of the energy which the substances concerned had in their original state, when uncombined, above that which they have after combination.

SCIENTIFIC WORK TO BE DONE.—Though a great deal of excellent work of this kind has already been done, the extent of the field hitherto investigated appears quite insignificant when we consider the boundless variety and complexity of the natural bodies with which we have to deal.

In fact the special work which lies before the physical inquirer, in the present state of science, is the determination of the quantity of energy which enters or leaves a material system during the passage of the system from its standard state to any other definite state.

HISTORY OF THE DOCTRINE OF ENERGY.—The scientific importance of giving a name to the quantity which we call kinetic energy seems to have been first recognized by Leibnitz, who gave to the product of the mass by the square of the velocity the name of *Vis Viva*. This is twice the kinetic energy.

Newton, in the "Scholium to the Laws of Motion," expresses the relation

between the rate at which work is done by the external agent, and the rate at which it is given out, stored up, or transformed by any machine or other material system, in the following statement, which he makes in order to show the wide extent of the application of the Third Law of Motion.

"If the action of the external agent is estimated by the product of its force into its velocity, and the reaction of the resistance in the same way by the product of the velocity of each part of the system into the resisting force arising from friction, cohesion, weight, and acceleration, the action and reaction will be equal to each other, whatever be the nature and motion of the system." That this statement of Newton's implicitly contains nearly the whole doctrine of energy was first pointed out by Thomson and Tait.

The words Action and Reaction as they occur in the enunciation of the Third Law of Motion are explained to mean Forces, that is to say, they are the

opposite aspects of one and the same Stress.

In the passage quoted above a new and different sense is given to these words by estimating Action and Reaction by the product of a force into the velocity of its point of application. According to this definition the Action of the external agent is the rate at which it does work. This is what is meant by the Power of a steam-engine or other prime mover. It is generally expressed by the estimated number of ideal horses which would be required to do the work at the same rate as the engine, and this is called the Horse-power of the engine.

When we wish to express by a single word the rate at which work is done by an agent we shall call it the Power of the agent, defining the power as the work done in the unit of time.

The use of the term Energy, in a precise and scientific sense, to express the quantity of work which a material sys-

tem can do, was introduced by Dr. Young.*

ON THE DIFFERENT FORMS OF ENERGY. —The energy which a body has in virtue of its motion is called kinetic energy.

A system may also have energy in virtue of its configuration, if the forces of the system are such that the system will do work against external resistance while it passes into another configuration. This energy is called Potential Energy. Thus when a stone has been lifted to a certain height above the earth's surface, the system of two bodies, the stone and the earth, has potential energy, and is able to do a certain amount of work during the descent of the stone. This potential energy is due to the fact that the stone and the earth attract each other, so that work has to be spent by the man who lifts the stone and draws it away from the earth, and after the stone is lifted the attraction between the earth and the stone is capable of doing

* "Lectures on Natural Philosophy," Lecture VIII.

work as the stone descends. This kind of energy, therefore, depends upon the work which the forces of the system would do if the parts of the system were to yield to the action of these forces. This is called the "Sum of the Tensions" by Helmholtz in his celebrated memoir on the "Conservation of Energy."* Thomson called it Statical Energy; it has also been called Energy of Position; but Rankine introduced the term Potential Energy—a very felicitous expression, since it not only signifies the energy which the system has not in actual possession, but only has the power to acquire, but it also indicates its connection with what has been called (on other grounds) the Potential Function.

The different forms in which energy has been found to exist in material systems have been placed in one or other of these two classes—Kinetic Energy, due to motion, and Potential Energy, due to configuration.

* Berlin, 1847. Translated in Taylor's "Scientific Memoirs," Feb. 1853.

Thus a hot body, by giving out heat to a colder body, may be made to do work by causing the cold body to expand in opposition to pressure. A material system, therefore, in which there is a non-uniform distribution of temperature has the capacity of doing work, or energy. This energy is now believed to be kinetic energy, due to a motion of agitation in the smallest parts of the hot body.

Gunpowder has energy, for when fired it is capable of setting a cannon-ball in motion. The energy of gunpowder is Chemical Energy, arising from the power which the constituents of gunpowder possess of arranging themselves in a new manner when exploded, so as to occupy a much larger volume than the gunpowder does. In the present state of science chemists figure to themselves chemical action as a rearrangement of particles under the action of forces tending to produce this change of arrangement. From this point of view, therefore, chemical energy is potential energy.

Air, compressed in the chamber of an air-gun, is capable of propelling a bullet. The energy of compressed air was at one time supposed to arise from the mutual repulsion of its particles. If this explanation were the true one its energy would be potential energy. In more recent times it has been thought that the particles of the air are in a state of motion, and that its pressure is caused by the impact of these particles on the sides of the vessel. According to this theory the energy of compressed air is kinetic energy.

There are thus many different modes in which a material system may possess energy, and it may be doubtful in some cases whether the energy is of the kinetic or the potential form. The nature of energy, however, is the same in whatever form it may be found. The quantity of energy can always be expressed as that of a body of a definite mass moving with a definite velocity.

CHAPTER VI.

RECAPITULATION.

RETROSPECT OF ABSTRACT DYNAMICS.—We have now gone through the part of the fundamental science of the motion of matter, which we have been able to treat in a manner sufficiently elementary to be consistent with the plan of this book.

It remains for us to take a general view of the relations between the parts of this science, and of the whole to other physical sciences, and this we can now do in a more satisfactory way than we could before we had entered into the subject.

KINEMATICS.—We began with kinematics, or the science of pure motion. In this division of the subject the ideas brought before us are those of space and time. The only attribute of matter which comes before us is its continuity of existence in space and time—the fact,

namely, that every particle of matter, at any instant of time, is in one place and in one only, and that its change of place during any interval of time is accomplished by moving along a continuous path.

Neither the force which affects the motion of the body, nor the mass of the body, on which the amount of force required to produce the motion depends, comes under our notice in the pure science of motion.

Force.—In the next division of the subject force is considered in the aspect of that which alters the motion of a mass.

If we confine our attention to a single body, our investigation enables us, from observation of its motion, to determine the direction and magnitude of the resultant force which acts on it, and this investigation is the exemplar and type of all researches undertaken for the purpose of the discovery and measurement of physical forces.

But this may be regarded as a mere application of the definition of a force, and not as a new physical truth.

It is when we come to define equal forces as those which produce equal rates of acceleration in the same mass, and equal masses are those which are equally accelerated by equal forces, that we find that these definitions of equality amount to the assertion of the physical truth, that the comparison of quantities of matter by the forces required to produce in them a given acceleration is a method which always leads to consistent results, whatever be the absolute values of the forces and the accelerations.

Stress.—The next step in the science of force is that in which we pass from the consideration of a force as acting on a body, to that of its being one aspect of that mutual action between two bodies, which is called by Newton Action and Reaction, and which is now more briefly expressed by the single word Stress.

RELATIVITY OF DYNAMICAL KNOWLEDGE.—Our whole progress up to this point may be described as a gradual development of the doctrine of relativity of all physical phenomena. Position we must evidently acknowledge to be relative, for we cannot describe the position of a body in any terms which do not express relation. The ordinary language about motion and rest does not so completely exclude the notion of their being measured absolutely, but the reason of this is, that in our ordinary language we tacitly assume that the earth is at rest.

As our ideas of space and motion become clearer, we come to see how the whole body of dynamical doctrine hangs together in one consistent system.

Our primitive notion may have been that to know absolutely where we are, and in what direction we are going, are essential elements of our knowledge as conscious beings.

But this notion, though undoubtedly held by many wise men in ancient times, has been gradually dispelled from the minds of students of physics.

There are no landmarks in space; one portion of space is exactly like every other portion, so that we cannot tell where we are. We are, as it were, on an unruffled sea, without stars, compass, soundings, wind, or tide, and we cannot tell in what direction we are going. We have no log which we can cast out to take a dead reckoning by; we may compute our rate of motion with respect to the neighboring bodies, but we do not know how these bodies may be moving in space.

Relativity of Force.—We cannot even tell what force may be acting on us; we can only tell the difference between the force acting on one thing and that acting on another.

We have an actual example of this in our every-day experience. The earth moves round the sun in a year at a distance of 91,520,000 miles, or 1.473×10^{13} centimeters. It follows from this that a force is exerted on the earth in the direction of the sun, which produces an ac-

celeration of the earth in the direction of the sun of about 0.019 in feet and seconds, or about $\frac{1}{1680}$ of the intensity of gravity at the earth's surface.

A force equal to the sixteen-hundredth part of the weight of a body might be easily measured by known experimental methods, especially if the direction of this force were differently inclined to the vertical at different hours of the day.

Now, if the attraction of the sun were exerted upon the solid part of the earth, as distinguished from the movable bodies on which we experiment, a body suspended by a string, and moving with the earth, would indicate the difference between the solar action on the body, and that on the earth as a whole.

If, for example, the sun attracted the earth and not the suspended body, then at sunrise the point of suspension, which is rigidly connected with the earth, would be drawn towards the sun, while the suspended body would be acted on only by the earth's attraction, and the

string would appear to be deflected away from the sun by a sixteen-hundredth part of the length of the string. At sunset the string would be deflected away from the setting sun by an equal amount; and as the sun sets at a different point of the compass from that at which he rises the deflections of the string would be in different directions, and the difference in the position of the plumb-line at sunrise and sunset would be easily observed.

But instead of this, the attraction of gravitation is exerted upon all kinds of matter equally at the same distance from the attracting body. At sunrise and sunset the center of the earth and the suspended body are nearly at the same distance from the sun, and no deflection of the plumb-line due to the sun's attraction can be observed at these times. The attraction of the sun, therefore, in so far as it is exerted equally upon all bodies on the earth, produces no effect on their relative motions. It is only the differences of the intensity and direction

of the attraction acting on different parts of the earth which can produce any effect, and these differences are so small for bodies at moderate distances that it is only when the body acted on is very large, as in the case of the ocean, that their effect becomes perceptible in the form of tides.

ROTATION.—In what we have hitherto said about the motion of bodies, we have tacitly assumed that, in comparing one configuration of the system with another, we are able to draw a line in the final configuration parallel to a line in the original configuration. In other words, we assume that there are certain directions in space which may be regarded as constant, and to which other directions may be referred during the motion of the system.

In astronomy, a line drawn from the earth to a star may be considered as fixed in direction, because the relative motion of the earth and the star is in general so small compared with the distance be-

tween them that the change of direction, even in a century, is very small. But it is manifest that all such directions of reference must be indicated by the configuration of a material system existing in space, and that if this system were altogether removed, the original directions of reference could never be recovered.

But, though it is impossible to determine the absolute velocity of a body in space, it is possible to determine whether the direction of a line in a material system is constant or variable.

For instance, it is possible by observations made on the earth alone, without reference to the heavenly bodies, to determine whether the earth is rotating or not.

So far as regards the geometrical configuration of the earth and the heavenly bodies, it is evidently all the same

"Whether the sun, predominant in heaven,
Rise on the earth, or earth rise on the sun;
He from the east his flaming road begin,
Or she from west her silent course advance
With inoffensive pace that spinning sleeps
On her soft axle, while she paces even,
And bears thee soft with the smooth air along."

The distances between the bodies composing the universe, whether celestial or terrestrial, and the angles between the lines joining them, are all that can be ascertained without an appeal to dynamical principles, and these will not be affected if any motion of rotation of the whole system, similar to that of a rigid body about an axis, is combined with the actual motion; so that from a geometrical point of view the Copernican system, according to which the earth rotates, has no advantage, except that of simplicity, over that in which the earth is supposed to be at rest, and the apparent motions of the heavenly bodies to be their absolute motions.

Even if we go a step further, and consider the dynamical theory of the earth rotating round its axis, we may account for its oblate figure, and for the equilibrium of the ocean and of all other bodies on its surface on either of two hypothesis—that of the motion of the earth round its axis, or that of the earth not rotating, but caused to assume its

oblate figure by a force acting outwards in all directions from its axis, the intensity of this force increasing as the distance from the axis increases. Such a force, if it acted on all kinds of matter alike, would account not only for the oblateness of the earth's figure, but for the conditions of equilibrium of all bodies at rest with respect to the earth.

It is only when we go further still, and consider the phenomena of bodies which are in motion with respect to the earth, that we are really constrained to admit that the earth rotates.

NEWTON'S DETERMINATION OF THE ABSOLUTE VELOCITY OF ROTATION.—Newton was the first to point out that the absolute motion of rotation of the earth might be demonstrated by experiments on the rotation of a material system. For instance, if a bucket of water is suspended from a beam by a string, and the string twisted so as to keep the bucket spinning round a vertical axis, the water will soon spin round at the

same rate as the bucket, so that the system of the water and the bucket turns round its axis like a solid body.

The water in the spinning bucket rises up at the sides and is depressed in the middle, showing that in order to make it move in a circle a pressure must be exerted towards the axis. This concavity of the surface depends on the absolute motion of rotation of the water and not on its relative rotation.

For instance, it does not depend on the rotation relative to the bucket. For at the beginning of the experiment, when we set the bucket spinning, and before the water has taken up the motion, the water and the bucket are in relative motion, but the surface of the water is flat, because the water is not rotating, but only the bucket.

When the water and the bucket rotate together, there is no motion of the one relative to the other, but the surface of the water is hollow, because it is rotating.

When the bucket is stopped, as long

as the water continues to rotate its surface remains hollow, showing that it is still rotating though the bucket is not.

It is manifestly the same, as regards this experiment, whether the rotation be in the direction of the hands of a watch or the opposite direction, provided the rate of rotation is the same.

Now let us suppose this experiment tried at the North Pole. Let the bucket be made, by a proper arrangement of clockwork, to rotate either in the direction of the hands of a watch, or in the opposite direction, at a perfectly regular rate.

If it is made to turn round by clockwork once in twenty-four hours (sidereal time) the way of the hands of a watch laid face upwards, it will be rotating as regards the earth, but not rotating as regards the stars.

If the clockwork is stopped, it will rotate with respect to the stars, but not with respect to the earth.

Finally, if it is made to turn round once in twenty-four hours (sidereal time)

in the opposite direction, it will be rotating with respect to the earth at the same rate as at first, but instead of being free from rotation as respects the stars, it will be rotating at the rate of one turn in twelve hours.

Hence if the earth is at rest, and the stars moving round it, the form of the surface will be the same in the first and last case; but if the earth is rotating, the water will be rotating in the last case but not in the first, and this will be made manifest by the water rising higher at the sides in the last case than in the first.

The surface of the water will not be really concave in any of the cases supposed, for the effect of gravity acting towards the center of the earth is to make the surface convex, as the surface of the sea is, and the rate of rotation in our experiment is not sufficiently rapid to make the surface concave. It will only make it slightly less convex than the surface of the sea in the last case, and slightly more convex in the first.

But the difference in the form of the surface of the water would be so exceedingly small, that with our methods of measurement it would be hopeless to attempt to determine the rotation of the earth in this way.

FOUCAULT'S PENDULUM.—The most satisfactory method of making an experiment for this purpose is that devised by M. Foucault.

A heavy ball is hung from a fixed point by a wire, so that it is capable of swinging like a pendulum in any vertical plane passing through the fixed point.

In starting the pendulum care must be taken that the wire, when at the lowest point of the swing, passes exactly through the position it assumes when hanging vertically. If it passes on one side of this position, it will return on the other side, and this motion of the pendulum round the vertical instead of through the vertical must be carefully avoided, because we wish to get rid of

all motions of rotation either in one direction or the other.

Let us consider the angular momentum of the pendulum about the vertical line through the fixed point.

At the instant at which the wire of the pendulum passes through the vertical line, the angular momentum about the vertical line is zero.

The force of gravity always acts parallel to this vertical line, so that it cannot produce angular momentum round it. The tension of the wire always acts through the fixed point, so that it cannot produce angular momentum about the vertical line.

Hence the pendulum can never acquire angular momentum about the vertical line through the point of suspension.

Hence when the wire is out of the vertical, the vertical plane through the center of the ball and the point of suspension cannot be rotating; for if it were, the pendulum would have an angular momentum about the vertical line.

Now let us suppose this experiment

performed at the North Pole. The plane of vibration of the pendulum will remain absolutely constant in direction, so that if the earth rotates the rotation of the earth will be made manifest.

We have only to draw a line on the earth parallel to the plane of vibration, and to compare the position of this line with that of the plane of vibration at a subsequent time.

As a pendulum of this kind properly suspended will swing for several hours, it is easy to ascertain whether the position of the plane of vibration is constant as regards the earth, as it would be if the earth is at rest, or constant as regards the stars, if the stars do not move round the earth.

We have supposed, for the sake of simplicity in the description, that the experiment is made at the North Pole. It is not necessary to go there in order to demonstrate the rotation of the earth.

The only region where the experiment will not show it is at the equator.

At every other place the pendulum

will indicate the rate of rotation of the earth with respect to the vertical line at that place. If at any instant the plane of the pendulum passes through a star near the horizon either rising or setting, it will continue to pass through that star as long as it is near the horizon. That is to say, the horizontal part of the apparent motion of a star on the horizon is equal to the rate of rotation of the plane of vibration of the pendulum.

It has been observed that the plane of vibration appears to rotate in the opposite direction in the southern hemisphere, and by a comparison of the rates at various places the actual time of rotation of the earth has been deduced without reference to astronomical observations. The mean value, as deduced from these experiments by Messrs. Galbraith and Houghton in their "Manual of Astronomy," is $23^{h.}\ 53^{m.}\ 37^{s.}$. The true time of rotation of the earth is $23^{h.}\ 56^{m}\ 4^{s.}$ mean solar time.

MATTER AND ENERGY.—All that we

know about matter relates to the series of phenomena in which energy is transferred from one portion of matter to another, till in some part of the series our bodies are affected, and we become conscious of a sensation.

By the mental process which is founded on such sensations we come to learn the conditions of these sensations, and to trace them to objects which are not part of ourselves, but in every case the fact that we learn is the mutual action between bodies. This mutual action we have endeavored to describe in this treatise. Under various aspects it is called Force, Action and Reaction, and Stress, and the evidence of it is the change of the motion of the bodies between which it acts.

The process by which stress produces change of motion is called Work, and, as we have already shown, work may be considered as the transference of Energy from one body or system to another.

Hence, as we have said, we are acquainted with matter only as that which

may have energy communicated to it from other matter, and which may, in its turn, communicate energy to other matter.

Energy, on the other hand, we know only as that which in all natural phenomena is continually passing from one portion of matter to another.

Test of a Material Substance.—Energy cannot exist except in connection with matter. Hence since, in the space between the sun and the earth, the luminous and thermal radiations, which have left the sun and which have not reached the earth, possess energy, the amount of which per cubic mile can be measured; this energy must belong to matter existing in the interplanetary spaces, and since it is only by the light which reaches us that we become aware of the existence of the most remote stars, we conclude that the matter which transmits light is disseminated through the whole of the visible universe.

ENERGY NOT CAPABLE OF IDENTIFICATION.—We cannot identify a particular portion of energy, or trace it through its transformations. It has no individual existence, such as that which we attribute to particular portions of matter.

The transactions of the material universe appear to be conducted, as it were, on a system of credit. Each transaction consists of the transfer of so much credit or energy from one body to another. This act of transfer or payment is called work. The energy so transferred does not retain any character by which it can be identified when it passes from one form to another.

ABSOLUTE VALUE OF THE ENERGY OF A BODY UNKNOWN.—The energy of a material system can only be estimated in a relative manner.

In the first place, though the energy of the motion of the parts relative to the center of mass of the system may be accurately defined, the whole energy consists of this together with the energy of

a mass equal to that of the whole system moving with the velocity of the center of mass. Now this latter velocity—that of the center of mass—can be estimated only with reference to some body external to the system, and the value which we assign to this velocity will be different according to the body which we select as our origin.

Hence the estimated kinetic energy of a material system contains a part, the value of which cannot be determined except by the arbitrary selection of an origin. The only origin which would not be arbitrary is the center of mass of the material universe, but this is a point the position and motion of which are quite unknown to us.

Latent Energy.—But the energy of a material system is indeterminate for another reason. We cannot reduce the system to a state in which it has no energy, and any energy which is never removed from the system must remain unperceived by us, for it is only as it en-

ters or leaves the system that we can take any account of it.

We must, therefore, regard the energy of a material system as a quantity of which we may ascertain the increase or diminution as the system passes from one definite condition to another. The absolute value of the energy in the standard condition is unknown to us, and it would be of no value to us if we did know it, as all phenomena depend on the variations of the energy, and not on its absolute value.

A Complete Discussion of Energy would include the whole of Physical Science.—The discussion of the various forms of energy—gravitational, electromagnetic, molecular, thermal, &c.—with the conditions of the transference of energy from one form to another, and the constant dissipation of the energy available for producing work, constitutes the whole of physical science, in so far as it has been developed in the dynamical form under the various designations

of Astronomy, Electricity, Magnetism, Optics, Theory of the Physical States of Bodies, Thermo-dynamics, and Chemistry.

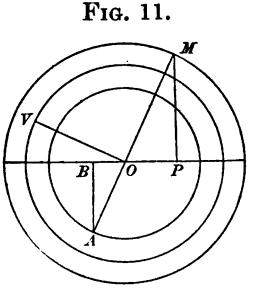
CHAPTER VII.

THE PENDULUM AND GRAVITY.

On Uniform Motion in a Circle.— Let M (Fig. 11) be a body moving in a circle with velocity V.

Let $\overline{O\,M}=r$ be the radius of the circle.

Fig. 11.



The direction of the velocity of M is that of the tangent to the circle. Draw

$\overline{O\,V}$ parallel to this direction through the center of the circle and equal to the distance described in unit of time with velocity V, then $\overline{O\,V} = V$.

If we take O as the origin of the diagram of velocity, V will represent the velocity of the body at M.

As the body moves round the circle, the point V will also describe a circle, and the velocity of the point V will be to that of M as $\overline{O\,V}$ to $\overline{O\,M}$.

If, therefore, we draw $\overline{O\,A}$ in $\overline{M\,O}$ produced, and therefore parallel to the direction of motion of V, and make $\overline{O\,A}$ a third proportional to $\overline{O\,M}$ and $\overline{O\,V}$, and if we assume O as the origin of the diagram of rate of acceleration, then the point A will represent the velocity of the point V, or, what is the same thing, the rate of acceleration of the point M.

Hence, when a body moves with uniform velocity in a circle, its acceleration is directed towards the center of the circle, and is a third proportional to the radius of the circle and the velocity of the body.

The orce acting on the body M is equal to the product of this acceleration into the mass of the body, or if F be this force

$$F=\frac{MV^2}{r}$$

Centrifugal Force.—This is the force which must act on the body M in order to keep it in the circle of radius v, in which it is moving with velocity V.

The direction of this force is towards the center of the circle.

If this force is applied by means of a string fastened to the body, the string will be in a state of tension. To a person holding the other end of the string this tension will appear to be directed towards the body M, as if the body M had a tendency to move away from the center of the circle which it is describing.

Hence this latter force is often called Centrifugal Force.

The force which really acts on the body, being directed towards the center

of the circle, is called Centripetal Force, and in some popular treatises the centripetal and centrifugal forces are described as opposing and balancing each other. But they are merely the different aspects of the same stress.

Periodic Time.—The time of describing the circumference of the circle is called the Periodic Time. If π represents the ratio of the circumference of a circle to its diameter, which is 3.14159 . . . . . the circumference of a circle of radius r is $2\pi r$, and since this is described in the periodic time T with velocity V, we have

$$2\pi r = V T$$

Hence $$F = 4\pi^2 M \frac{r}{T^2}$$

The rate of circular motion is often expressed by the number of revolutions in unit of time. Let this number be denoted by n, then

$$n T = 1$$

and $$F = 4\pi^2 M r n^2.$$

On Simple Harmonic Vibrations.—If while the body M (Fig. 11) moves in a circle with uniform velocity another point P moves in a fixed diameter of the circle, so as to be always at the foot of the perpendicular from M on that diameter, the body P is said to execute Simple Harmonic Vibrations.

The radius, r, of the circle is called the Amplitude of the vibration.

The periodic time of M is called the Periodic Time of Vibration.

The angle which O M makes with the positive direction of the fixed diameter is called the Phase of the vibration.

On the Force Acting on the Vibrating Body.—The only difference between the motions of M and P is that M has a vertical motion compounded with a horizontal motion which is the same as that of P. Hence the velocity and the acceleration of the two bodies differ only with respect to the vertical part of the velocity and acceleration of M.

The acceleration of P is therefore the horizontal component of that of M, and since the acceleration of M is represented by O A, which is in the direction of M O produced, the acceleration of P will be represented by O B, where B is the foot of the perpendicular from A on the horizontal diameter. Now by similar triangles O M P, O A B

$$OM : OA :: OP : OB$$

But $OM = r$ and $OA = -4\pi^2 \frac{r}{T^2}$. Hence

$$OB = -\frac{4\pi^2}{T^2} OP = -4\pi^2 n^2 OP$$

In simple harmonic vibration, therefore, the acceleration is always directed towards the center of vibration, and is equal to the distance from that center multiplied by $4\pi^4 n^2$, and if the mass of the vibrating body is P, the force acting on it at a distance x from O is $4\pi^2 n^2 P x$.

It appears, therefore, that a body which executes simple harmonic vibrations in a straight line is acted on by a force which varies as the distance from

the center of vibration, and the value of this force at a given distance depends only on that distance, on the mass of the body, and on the square of the number of vibrations in unit of time, and is independent of the amplitude of the vibrations.

Isochronous Vibrations.—It follows from this that if a body moves in a straight line and is acted on by a force directed towards a fixed point on the line and varying as the distance from that point, it will execute simple harmonic vibrations, the periodic time of which will be the same whatever the amplitude of vibration.

If, for a particular kind of displacement of a body, as turning round an axis, the force tending to bring it back to a given position varies as the displacement, the body will execute simple harmonic vibrations about that position, the periodic time of which will be independent of their amplitude.

Vibrations of this kind, which are ex-

ecuted in the same time whatever be their amplitude, are called Isochronous Vibrations.

Potential Energy of the Vibrating Body.—The velocity of the body when it passes through the point of equilibrium is equal to that of the body moving in the circle, or $V=2\pi r n$,

where r is the amplitude of vibration and n is the number of double vibrations per second.

Hence the kinetic energy of the vibrating body at the point of equilibrium is

$$\tfrac{1}{2} M V^2=2\pi^2 M r^2 n^2$$

where M is the mass of the body.

At the extreme elongation, where $x=r$, the velocity, and therefore the kinetic energy, of the body is zero. The diminution of kinetic energy must correspond to an equal increase of potential energy. Hence if we reckon the potential energy from the configuration in which the body is at its point of equili-

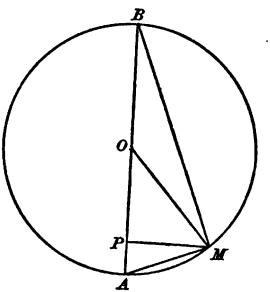
brium, its potential energy when at a distance, r, from this point is $2\pi^2 M n^2 r^2$.

This is the potential energy of a body which vibrates isochronously, and executes n double variations per second when it is at rest at the distance, r, from the point of equilibrium. As the potential energy does not depend on the motion of the body, but only on its position, we may write it $2\pi^2 M n^2 x^2$,
where x is the distance from the point of equilibrium.

The Simple Pendulum.—The simple pendulum consists of a small heavy body called the bob, suspended from a fixed point by a fine string of invariable length. The bob is supposed to be so small that its motion may be treated as that of a material particle, and the string is supposed to be so fine that we may neglect its mass and weight. The bob is set in motion so as to swing through a small angle in a vertical plane. Its path, therefore, is an arc of a circle, whose center is the point of suspension,

O, and whose radius is the length of the string, which we shall denote by l.

Fig. 12.



Let O (Fig. 12) be the point of suspension and O A the position of the pendulum when hanging vertically. When the bob is at M it is higher than when it is at A by the height $AP = \frac{\overline{AM}^2}{\overline{AB}}$ where AM is the chord of the arc ALM and $AB = 2l$.

If M be the mass of the bob and g the intensity of gravity the weight of the bob will be Mg and the work done against gravity during the motion of the

bob from A to M will be $Mg\,\overline{AP}$. This, therefore, is the potential energy of the pendulum when the bob is at M, reckoning the energy zero when the bob is at A.

We may write this energy

$$\frac{Mg}{2l}\,\overline{AM}^2$$

The potential energy of the bob when displaced through any arc varies as the square of the chord of that arc.

If it had varied as the square of the arc itself in which the bob moves, the vibrations would have been strictly isochronous. As the potential energy varies more slowly than the square of the arc, the period of each vibration will be greater when the amplitude is greater.

For very small vibrations, however, we may neglect the difference between the chord and the arc, and denoting the arc by x we may write the potential energy

$$\frac{Mg}{2l}x^2$$

But we have already shown that in harmonic vibrations the potential energy is $2\pi^2 M n^2 x^2$.

Equating these two expressions and clearing fractions we find

$$g = 4\pi^2 n^2 l,$$

where g is the intensity of gravity, π is the ratio of the circumference of a circle to its diameter, n is the number of vibrations of the pendulum in unit of time, and l is the length of the pendulum.

A Rigid Pendulum.—If we could construct a pendulum with a bob so small and a string so fine that it might be regarded for practical purposes as a simple pendulum, it would be easy to determine g by this method. But all real pendulums have bobs of considerable size, and in order to preserve the length invariable the bob must be connected with the point of suspension by a stout rod, the mass of which cannot be neglected. It is always possible, however, to determine the length of a simple pendulum whose vibrations would be

executed in the same manner as those of a pendulum of any shape.

The complete discussion of this subject would lead us into calculations beyond the limits of this treatise. We may, however, arrive at the most important result without calculation as follows:

The motion of a rigid body in one plane may be completely defined by stating the motion of its center of mass, and the motion of the body round its center of mass.

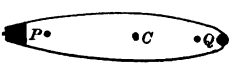
The force required to produce a given change in the motion of the center of mass depends only on the mass of the body. "Effect of External Forces on the Motion of the Center of Mass."

The moment required to produce a given change of angular velocity about the center of mass depends on the distribution of the mass, being greater the further the different parts of the body are from the center of mass.

If, therefore, we form a system of two particles rigidly connected, the sum of

the masses being equal to the mass of a pendulum, their center of mass coinciding with that of the pendulum, and their distances from the center of mass being such that a couple of the same moment is required to produce a given rotatory motion about the center of mass of the new system as about that of the pendulum, then the new system will for motions in a center plane be dynamically equivalent to the given pendulum, that is, if the two systems are moved in the same way the forces required to guide the motion will be equal. Since the two particles may have any ratio, provided the sum of their masses is equal to the mass of the pendulum, and since the line joining them may have any direction provided it passes through the center of mass, we may arrange them so that one of the particles corresponds to any given point of the pendulum, say, the point of suspension P (Fig. 13). The mass of

FIG. 13.



this particle and the position and mass of the other at Q will be determinate. The position of the second particle, Q, is called the Center of Oscillation. Now in the system of two particles, if one of them, P, is fixed, and the other, Q, allowed to swing under the action of gravity, we have a simple pendulum. For one of the particles, P, acts as the point of suspension, and the other, Q, is at an invariable distance from it, so that the connection between them is the same as if they were united by a string of length $l = \overline{PQ}$.

Hence a pendulum of any form swings in exactly the same manner as a simple pendulum whose length is the distance from the center of suspension to the center of oscillation.

Inversion of the Pendulum.—Now let us suppose the system of two particles inverted, Q being made the point of suspension and P being made to swing. We have now a simple pendulum of the same length as before. Its vibrations

will therefore be executed in the same time. But it is dynamically equivalent to the pendulum suspended by its center of oscillation.

Hence if a pendulum be inverted and suspended by its center of oscillation its vibrations will have the same period as before, and the distance between the center of suspension and that of oscillation will be equal to that of a simple pendulum having the same time of vibration.

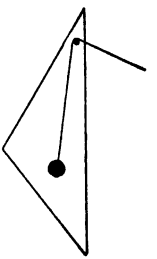
It was in this way that Captain Kater determined the length of the simple pendulum which vibrates seconds.

He constructed a pendulum which could be made to vibrate about two knife edges, on opposite sides of the center of mass and at *unequal* distances from it.

By certain adjustments, he made the time of vibration the same whether the one knife edge or the other were the center of suspension. The length of the corresponding simple pendulum was then found by measuring the distance between the knife edges.

ILLUSTRATION OF KATER'S PENDULUM. —The principle of Kater's Pendulum may be illustrated by a very simple and striking experiment. Take a flat board of any form (Fig. 14), and drive a piece

FIG. 14.



of wire through it near its edge, and allow it to hang in a vertical plane, holding the ends of the wire by the finger and thumb. Take a small bullet, fasten it to the end of a thread and allow the thread to pass over the wire, so that the bullet hangs close to the board. Move the hand by which you hold the wire horizontally in the plane of the board, and observe whether the board moves forwards or backwards with respect to

the bullet. If it moves forwards lengthen the string, if backwards shorten it till the bullet and the board move together. Now mark the point of the board opposite the center of the bullet and fasten the string to the wire. You will find that if you hold the wire by the ends and move it in any manner, however sudden and irregular, in the plane of the board, the bullet will never quit the marked spot on the board.

Hence this spot is called the center of oscillation, because when the board is oscillating about the wire when fixed it oscillates as if it consisted of a single particle placed at the spot.

It is also called the center of percussion, because if the board is at rest and the wire is suddenly moved horizontally the board will at first begin to rotate about the spot as a center.

DETERMINATION OF THE INTENSITY OF GRAVITY.—The most direct method of determining g is, no doubt, to let a body fall and find what velocity it has gained

in a second, but it is very difficult to make accurate observations of the motion of bodies when their velocities are so great as 981 centimeters per second, and besides, the experiment would have to be conducted in a vessel from which the air has been exhausted, as the resistance of the air to such rapid motion is very considerable, compared with the weight of the falling body.

The experiment with the pendulum is much more satisfactory. By making the arc of vibration very small, the motion of the bob becomes so slow that the resistance of the air can have very little influence on the time of vibration. In the best experiments the pendulum is swung in an air-tight vessel from which the air is exhausted.

Besides this, the motion repeats itself, and the pendulum swings to and fro hundreds, or even thousands, of times before the various resistances to which it is exposed reduce the amplitude of the vibrations till they can no longer be observed.

Thus the actual observation consists not in watching the beginning and end of one vibration, but in determining the duration of a series of many hundred vibrations, and thence deducing the time of a single vibration.

The observer is relieved from the labor of counting the whole number of vibrations, and the measurement is made one of the most accurate in the whole range of practical science by the following method:

METHOD OF OBSERVATION.—A pendulum clock is placed behind the experimental pendulum, so that when both pendulums are hanging vertically the bob, or some other part of the experimental pendulum, just hides a white spot on the clock pendulum, as seen by a telescope fixed at some distance in front of the clock.

Observations of the transit of "clock stars" across the meridian are made from time to time, and from these the rate of the clock is deduced in terms of "mean solar time."

The experimental pendulum is then set a swinging, and the two pendulums are observed through the telescope. Let us suppose that the time of a single vibration is not exactly that of the clock pendulum, but a little more.

The observer at the telescope sees the clock pendulum always gaining on the experimental pendulum, till at last the experimental pendulum just hides the white spot on the clock pendulum as it crosses the vertical line. The time at which this takes place is observed and recorded as the First Positive Coincidence.

The clock pendulum continues to gain on the other, and after a certain time the two pendulums cross the vertical line at the same instant in opposite directions. The time of this is recorded as the First Negative Coincidence. After an equal interval of time there will be a second positive coincidence, and so on.

By this method the clock itself counts the number, N, of vibrations of its own

pendulum between the coincidences. During this time the experimental pendulum has executed one vibration less than the clock. Hence the time of vibration of the experimental pendulum is $\frac{N}{N-1}$ seconds of clock time.

When there is no exact coincidence, but when the clock pendulum is ahead of the experimental pendulum at one passage of the vertical and behind at the next, a little practice on the part of the observer will enable him to estimate at what time between the passages the two pendulums must have been in the same phase. The epoch of coincidence can thus be estimated to a fraction of a second.

ESTIMATION OF ERROR.—The experimental pendulum will go on swinging for some hours, so that the whole time to be measured may be ten thousand or more vibrations.

But the error introduced into the calculated time of vibration, by a mistake

even of a whole second in noting the time of vibration, may be made exceedingly small by prolonging the experiment.

For if we observe the first and the nth coincidence, and find that they are separated by an interval of N seconds of the clock, the experimental pendulum will have lost n vibrations, as compared with the clock, and will have made $N-n$ vibrations in N seconds. Hence the time of a single vibration is $T=\frac{N}{N-n}$ seconds of clock time.

Let us suppose, however, that by a mistake of a second we note down the last coincidence as taking place $N+1$ seconds after the first. The value of T as deduced from this result would be

$$T'=\frac{N+1}{N+1-n}$$

and the error introduced by the mistake of a second will be

$$T'-T=\frac{N+1}{N+1-n}-\frac{N}{N-n}$$

$$= \frac{n}{(N+1-n)\ (N-n)}$$

If N is 10000 and n is 100, a mistake of one second in noting the time of coincidence will alter the value of T only about one-millionth part of its value.

CHAPTER VIII.

UNIVERSAL GRAVITATION.

NEWTON'S METHOD.—The most instructive example of the method of dynamical reasoning is that by which Newton determined the law of the force with which the heavenly bodies act on each other.

The process of dynamical reasoning consists in deducing from the successive configurations of the heavenly bodies, as observed by astronomers, their velocities and their accelerations, and in this way determining the direction and the relative magnitude of the force which acts on them.

Kepler had already prepared the way for Newton's investigation, by deducing from a careful study of the observations of Tycho Brahe the three laws of planetary motion which bear his name.

KEPLER'S LAWS.—Kepler's Laws are purely kinematical. They completely describe the motions of the planets, but they say nothing about the forces by which these motions are determined.

Their dynamical interpretation was discovered by Newton.

The first and second law relate to the motion of a single planet.

Law I.—The areas swept out by the vector drawn from the sun to a planet are proportional to the times of describing them. If h denotes twice the area swept out in unit of time, twice the area swept out in time t will be $h\,t$, and if P is the mass of the planet, $\mathrm{P}\,h\,t$ will be the mass-area as defined in "Definition of a Mass-Area." Hence the angular momentum of the planet about the sun, which is the rate of change of the

mass-area, will be P h, a constant quantity.

Hence, by "Moment of a Force about a Point," the force, if any, which acts on the planet must have no moment with respect to the sun, for if it had it would increase or diminish the angular momentum at a rate measured by the value of this moment.

Hence, whatever be the force which acts on the planet, the direction of this force must always pass through the sun.

Angular Velocity.—Definition. The angular velocity of a vector is the rate at which the angle increases which it makes with a fixed vector in the plane of its motion.

If ω is the angular velocity of a vector, and r its length, the rate at which it sweeps out an area is $\frac{1}{2}\omega r^2$. Hence,

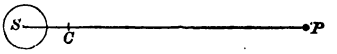
$$h = \omega r^2$$

and since h is constant, ω, the angular velocity of a planet's motion round the sun, varies inversely as the square of the distance from the sun.

This is true whatever the law of force may be, provided the force acting on the planet always passes through the sun.

MOTION ABOUT THE CENTER OF MASS. —Since the stress between the planet and the sun acts on both bodies, neither of them can remain at rest. The only point whose motion is not affected by the stress is the center of mass of the two bodies.

FIG. 15.



If r is the distance S P (Fig. 15), and if C is the center of mass, $\overline{SC}=\frac{Pr}{S+P}$ and $\overline{CP}=\frac{Sr}{S+P}$. The angular momentum of P about C is $P\omega\frac{S^2r^2}{(S+P)^2}=\frac{PS^2}{(S+P)^2h}$.

THE ORBIT.—We have already made use of diagrams of configuration and of velocity in studying the motion of a material system. These diagrams, however, represent only the state of the system at a given instant; and this state is indicated by the relative position of points corresponding to the bodies forming the system.

It is often, however, convenient to represent in a single diagram the whole series of configurations or velocities which the system assumes. If we suppose the points of the diagram to move so as continually to represent the state of the moving system, each point of the diagram will trace out a line, straight or curved.

On the diagram of configuration, this line is called, in general, the Path of the body. In the case of the heavenly bodies it is often called the Orbit.

THE HODOGRAPH.—On the diagram of velocity the line traced out by each moving point is called the Hodograph of the body to which it corresponds.

The study of the Hodograph, as a method of investigating the motion of a body, was introduced by Sir W. R. Hamilton. The hodograph may be defined as the path traced out by the extremity of a vector which continually represents, in direction and magnitude, the velocity of a moving body.

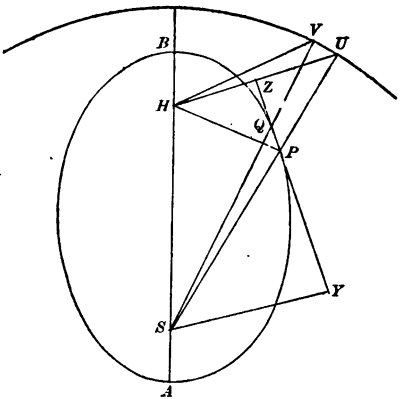
In applying the method of the hodograph to a planet, the orbit of which is in one plane, we shall find it convenient to suppose the hodograph turned round its origin through a right angle, so that the vector of the hodograph is perpendicular instead of parallel to the velocity it represents.

KEPLER'S SECOND LAW.—LAW II.—The orbit of a planet with respect to the sun is an ellipse, the sun being in one of the foci.

Let A P Q B (Fig. 16) be the elliptic orbit. Let S be the sun in one focus, and let H be the other focus. Produce S P to U, so that S U is equal to the transverse axis A B, and join H U, then

H U will be proportional and perpendicular to the velocity at P.

FIG. 16.



For bisect H U in Z and join Z P, Z P will be a tangent to the ellipse at P; let S Y be a perpendicular from S on this tangent.

If v is the velocity at P, and h twice the area swept out in unit of time $h=v\,\overline{SY}$.

Also if b is half the conjugate axis of the ellipse

$$\overline{SY}\cdot\overline{HZ}=b^2$$

Now HU = 2HZ; hence

$$v = \frac{1}{2}\frac{h}{b^2}\overline{\mathrm{HU}}.$$

Hence HU is always proportional to the velocity, and it is perpendicular to its direction. Now SU is always equal to AB. Hence the circle whose center is S and radius A B is the hodograph of the planet, H being the origin of the hodograph.

The corresponding points of the orbit and the hodograph are those which lie in the same straight line through S.

Thus P corresponds to U and Q to V.

The velocity communicated to the body during its passage from P to Q is represented by the geometrical difference between the vectors HU and HV, that is, by the line UV, and it is perpendicular to this arc of the circle, and is therefore, as we have already proved, directed towards S.

If P Q is the arc described in unit of time, then UV represents the acceleration, and since UV is on a circle whose center is S, UV will be a measure of the

angular velocity of the planet about S. Hence the acceleration is proportional to the angular velocity, and this by "Angular Velocity" is inversely as the square of the distance SP. Hence the acceleration of the planet is in the direction of the sun, and is inversely as the square of the distance from the sun.

This, therefore, is the law according to which the attraction of the sun on a planet varies as the planet moves in its orbit and alters its distance from the sun.

FORCE ON A PLANET.—Since, as we have already shown, the orbit of the planet with respect to the center of mass of the sun and planet has its dimensions in the ratio of S to S+P to those of the orbit of the planet with respect to the Sun, if $2a$ and $2b$ are the axes of the orbit of the planet with respect to the sun, the area is $\pi\ a\ b$, and if T is the time of going completely round the orbit, the value of h is $2\pi \frac{ab}{T}$

The velocity with respect to the sun is, therefore,

$$\pi \frac{a}{T\,b} \overline{HU}$$

With respect to the center of mass it is

$$\frac{S}{S+P} \frac{\pi a}{T b} \overline{HU}$$

The acceleration of the planet towards the center of mass is

$$\frac{S}{S+P} \frac{\pi a}{T b} U\,V$$

and the impulse on that planet whose mass is P is therefore

$$\frac{S\,.\,P}{S+P} \frac{\pi a}{T b} U\,V$$

Let t be the time of describing P Q, then twice the area S P Q is

$$h\,t = \omega\,r^2\,t$$

and $$UV = 2a\,\omega t = 2a\,\frac{h}{r^2}\,t = 4\pi\,\frac{a^3 b}{T r^3} t.$$

Hence the force on the planet is

$$F = 4\pi^2\,\frac{S\,.\,P}{S+P}\,\frac{a^3}{T^2 r^2}$$

This then is the value of the stress or

attraction between a planet and the sun in terms of their masses P and S, their mean distance a, their actual distance r, and the periodical time T.

INTERPRETATION OF KEPLER'S THIRD LAW.—To compare the attraction between the sun and different planets, Newton made use of Kepler's third law.

Law III.—The squares of the time of different planets are proportional to the cubes of their mean distances.

In other words $\frac{a^3}{T^2}$ is a constant, say $\frac{C}{4\pi^2}$

Hence $$F = C\frac{S.P}{S+P}\frac{1}{r^2}$$

In the case of the smaller planets their masses are so small, compared with that of the sun, that $\frac{S}{S+P}$ may be put equal to 1, so that $F = C\,P\,\frac{1}{r^2}$

or the attraction on a planet is proportional to its mass and inversely as the square of its distance.

LAW OF GRAVITATION.—This is the

most remarkable fact about the attraction of gravitation, that at the same distance it acts equally on equal masses of substances of all kinds. This is proved by pendulum experiments for the different kinds of matter at the surface of the earth. Newton extended the law to the matter of which the different planets are composed.

It had been suggested, before Newton proved it, that the sun as a whole attracts a planet as a whole, and the law of the inverse square had also been previously stated, but in the hands of Newton the doctrine of gravitation assumed its final form.

Every portion of matter attracts every other portion of matter, and the stress between them is proportional to the product of their masses divided by the square of their distance.

For if the attraction between a gramme of matter in the sun and a gramme of matter in a planet at distance r is $\frac{C}{r^2}$ where C is a constant, then if there are

S grammes in the sun and P in the planet the whole attraction between the sun and one gramme in the planet will be $\frac{CS}{r^2}$, and the whole attraction between the sun and the planet will be $C\frac{SP}{r^2}$.

Comparing this statement of Newton's "Law of Universal Gravitation" with the value of F formerly obtained we find

$$C\frac{S.P}{r^2}=4\pi^2\frac{S.P}{S+P}\frac{a^3}{T^2r^2}$$

$$\text{or } 4\pi^2 a^3=C(S+P)T^2.$$

AMENDED FORM OF KEPLER'S THIRD LAW.—Hence Kepler's Third Law must be amended thus:

The cubes of the mean distances are as the squares of the times multiplied into the sum of the masses of the sun and the planet.

In the case of the larger planets, Jupiter, Saturn, &c., the value of S+P is considerably greater than in the case of the earth and the smaller planets.

Hence the periodic times of the larger planets should be somewhat less than they would be according to Kepler's law, and this is found to be the case.

In the following table the mean distances (a) of the planets are given in terms of the mean distance of the earth, and the periodic time T in terms of the sidereal year:

(*See Table on following page.*)

It appears from the table that Kepler's third law is very nearly accurate, for a^3 is very nearly equal to T^2, but that for those planets whose mass is less than that of the earth—namely, Mercury, Venus and Mars—a^3 is less than T^2, whereas for Jupiter, Saturn, Uranus and Neptune, whose mass is greater than that of the earth, a^3 is greater than T^2.

Potential Energy due to Gravitation.—The potential energy of the gravitation between the bodies S and P may be calculated when we know the attraction between them in terms of their distance. The process of calculation by

Planet.	a	T	a_2	T_2	a^2-T_2
Mercury..	0.387098	0.24084	0.0580046	0.0580049	− 0.0000003
Venus....	0.72333	0.61518	0.378451	0.378453	− 0.0000002
Earth.....	1.0000	1.00000	1.00000	1.00000	
Mars.....	1.52369	1.88082	3.53746	3.53747	− 0.00001
Jupiter....	5.20278	11.8618	140.832	140.701	+ 0.131
Saturn....	9.53879	29.4560	867.914	867.658	+ 0.256
Uranus...	19.1824	84.0123	7058.44	7058.07	+ 0.37
Neptune..	30.037	164.616	27100.0	27098.4	+ 1.6

which we sum up the effects of a continually varying quantity belongs to the Integral Calculus, and though in this case the calculation may be explained by elementary methods, we shall rather deduce the potential energy directly from Kepler's first and second laws.

These laws completely define the motion of the sun and planet, and therefore we may find the kinetic energy of the system corresponding to any part of the elliptic orbit. Now, since the sun and planet form a conservative system, the sum of the kinetic and potential energy is constant, and therefore when we know the kinetic energy we may deduce that part of the potential energy which depends on the distance between the bodies.

KINETIC ENERGY OF THE SYSTEM.—To determine the kinetic energy we observe that the velocity of the planet with respect to the sun is by "Kepler's Second Law."

$$v = \tfrac{1}{2}\frac{h}{b'}\,\overline{\mathrm{HU}}$$

The velocities of the planet and the sun with respect to the center of mass of the system are respectively

$$\frac{S}{S+P}\,v \quad \text{and} \quad \frac{P}{S+P}\,v$$

The kinetic energies of the planet and the sun are therefore

$$\tfrac{1}{2}P\frac{S^2v^2}{(S+P)^2} \quad \text{and} \quad \tfrac{1}{2}S\frac{P^2}{(S+P^2)}\,v^2$$

and the whole kinetic energy is

$$\tfrac{1}{2}\frac{S\,.\,P}{S+P}\,v^2=\tfrac{1}{4}\,\frac{S\,.\,P}{S+P}\,\frac{h^2}{b^4}\,\overline{HU}^2$$

To determine v^2 in terms of $\overline{SP}$ or r, we observe that by the law of areas

$$v\,.\,\overline{SY}=h=\frac{2\pi ab}{T} \quad . \quad . \quad . \quad (1)$$

also by a property of the ellipse

$$\overline{HZ}\,.\,\overline{SY}=b^2 \quad . \quad . \quad . \quad . \quad (2)$$

and by the similar triangles HZP and SYP

$$\frac{\overline{SY}}{\overline{HZ}}=\frac{\overline{HP}}{\overline{SP}}=\frac{r}{2a-r} \quad . \quad . \quad (3)$$

multiplying (2) and (3) we find

$$\overline{SY}^2 = \frac{b^2 r}{2a - r}$$

Hence by (1)

$$v^2 = \frac{4\pi^2 a^2 b^2}{T^2} \frac{1}{\overline{SY}^2} = \frac{4\pi^2 a^2}{T^2} \left(\frac{2a}{r} - 1\right)$$

and the kinetic energy of the system is

$$\frac{4\pi^2 a^2}{T^2} \frac{S \cdot P}{S+P} \left(\frac{1}{r} - \frac{1}{2a}\right)$$

and this by the equation at the end of "Law of Gravitation" becomes

$$C \cdot S \cdot P \left(\frac{1}{r} - \frac{1}{2a}\right)$$

where C is the constant of gravitation.

This is the value of the kinetic energy of the two bodies S and P when moving in an ellipse of which the transverse axis is $2a$.

POTENTIAL ENERGY OF THE SYSTEM.—The sum of the kinetic and potential energies is constant, but its absolute value is by "Absolute Value of the Energy of a Body Unknown," and not necessary to be known.

Hence if we assume that the potential energy is of the form

$$K - C \cdot S \cdot P \frac{1}{r}$$

the second term, which is the only one depending on the distance, r, is also the only one which we have anything to do with. The other term K represents the work done by gravitation while the two bodies, originally at an infinite distance from each other, are allowed to approach as near as their dimensions will allow them.

The Moon is a Heavy Body.—Having thus determined the law of the force between each planet and the sun, Newton proceeded to show that the observed weight of bodies at the earth's surface and the force which retains the moon in her orbit round the earth are related to each other according to the same law of the inverse square of the distance.

This force of gravity acts in every region accessible to us, at the top of the highest mountains and at the highest point reached by balloons. Its intensity, as measured by pendulum experiments,

decreases as we ascend; and although the height to which we can ascend is so small, compared with the earth's radius, that we cannot from observations of this kind infer that gravity varies inversely as the square of the distance from the center of the earth, the observed decrease of the intensity of gravity is consistent with this law, the form of which had been suggested to Newton by the motion of the planets.

Assuming, then, that the intensity of gravity varies inversely as the square of the distance from the center of the earth, and knowing its value at the surface of the earth, Newton calculated its value at the mean distance of the moon.

His first calculations were vitiated by his adopting an erroneous estimate of the dimensions of the earth. When, however, he had obtained a more correct value of this quantity he found that the intensity of gravity, calculated for a distance equal to that of the moon, was equal to the force required to keep the moon in her orbit. He thus identified

the force which acts between the earth and the moon with that which causes bodies near the earth's surface to fall towards the earth.

CAVENDISH'S EXPERIMENT. -- Having thus shown that the force with which the heavenly bodies attract each other is of the same kind as that with which bodies that we can handle are attracted to the earth, it remained to be shown that bodies such as we can handle attract one another.

The difficulty of doing this arises from the fact that the mass of bodies which we can handle is so small compared with that of the earth, that even when we bring the two bodies as near as we can the attraction between them is an exceedingly small fraction of the weight of either.

We cannot get rid of the attraction of the earth, but we must arrange the experiment in such a way that it interferes as little as is possible with the effects of the attraction of the other body.

The apparatus devised by the Rev. John Michell for this purpose was that which has since received the name of the Torsion Balance. Michell died before he was able to make the experiment, but his apparatus afterwards came into the hands of Henry Cavendish, who improved it in many respects, and measured the attraction between large leaden balls and small balls suspended from the arms of the balance. A similar instrument was afterwards independently invented by Coulomb for measuring small electric and magnetic forces, and it continues to be the best instrument known to science for the measurement of small forces of all kinds.

THE TORSION BALANCE.—The Torsion Balance consists of a horizontal rod suspended by a wire from a fixed support. When the rod is turned round by an external force in a horizontal plane it twists the wire, and the wire, being elastic, tends to resist this strain and to untwist itself. This force of torsion is

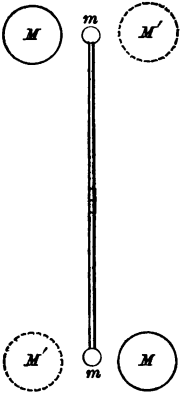
proportional to the angle through which the wire is twisted, so that if we cause a force to act in a horizontal direction at right angles to the rod at its extremity, we may, by observing the angle through which the force is able to turn the rod, determine the magnitude of the force.

The force is proportional to the angle of torsion and to the fourth power of the diameter of the wire, and inversely to the length of the rod and the length of the wire.

Hence, by using a long fine wire and a long rod, we may measure very small forces.

In the experiment of Cavendish two spheres of equal mass, m, are suspended from the extremities of the rod of the torsion balance. We shall for the present neglect the mass of the rod in comparison with that of the spheres. Two larger spheres of equal mass, M, are so arranged that they can be placed either at M and M or at M′ and M′. In the former position they tend by their attraction on the smaller spheres, m and

FIG. 17.



m, to turn the rod of the balance in the direction of the arrrows. In the latter position they tend to turn it in the opposite direction. The torsion balance and its suspended spheres are enclosed in a case, to prevent their being disturbed by currents of air. The position of the rod of the balance is ascertained by observing a graduated scale as seen by reflection in a vertical mirror fastened to the middle of the rod. The balance is placed in a room by itself, and the observer

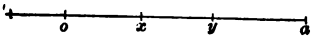
does not enter the room, but observes the image of the graduated scale with a telescope.

METHOD OF THE EXPERIMENT.—The time, T, of a double vibration of the torsion balance is first ascertained, and also the position of equilibrium of the centers of the suspended spheres.

The large spheres are then brought up to the positions MM, so that the center of each is at a distance from the position of equilibrium of the center of the suspended sphere.

No attempt is made to wait till the vibrations of the beam have subsided, but the scale-divisions corresponding to the extremities of a single vibration are observed, and are found to be distant x and y respectively from the position of equilibrium. At these points the rod is, for an instant, at rest, so that its energy is entirely potential, and since the total energy is constant, the potential energy corresponding to the position x must be equal to that corresponding to the position y.

FIG. 18.



Now if T be the time of a double vibration about the point of equilibrium O, the potential energy due to torsion when the scale reading is x is by "Potential Energy of the Vibrating Body"

$$\frac{2\pi^2 m}{T^2} x^2$$

and that due to the gravitation between m and M is by "Potential Energy of the System"

$$K - C \frac{m\,M}{a-x}$$

The potential energy of the whole system in the position x is therefore

$$K - C \frac{m\,M}{a-x} + \frac{2\pi^2 m}{T^2} x^2$$

In the position y it is

$$K - C \frac{m\,M}{a-y} + \frac{2\pi^2 m}{T^2} y^2$$

and since the potential energy in these two positions is equal,

$$C\,m\,M\left(\frac{1}{a-y} - \frac{1}{a-x}\right) = \frac{2\pi^2 m}{T^2}(y^2 - x^2)$$

Hence

$$C=\frac{2\pi^2}{MT^2}(x+y)(a-x)(a-y)$$

By this equation C, the constant of gravitation, is determined in terms of the observed quantities, M the mass of the large spheres in grammes, T the time of a double vibration in seconds, and the distances x y and a in centimeters.

According to Baily's experiments, $C=6.5\times10^{-8}$. If we assume the unit of mass, so that at a distance unity it would produce an acceleration unity, the centimeter and the second being units, the unit of mass would be about 1.537×10^{7} grammes, or 15.37 tonnes. This unit of mass reduces C, the constant of gravitation, to unity. It is therefore used in the calculations of physical astronomy.

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Since, therefore, we find in so great a number of cases occurring in regions remote from each other that the force of gravitation depends on the mass of bodies only, and not on their chemical nature or physical state, we are led to

conclude that this is true for all substances.

For instance, no man of science doubts that two portions of atmospheric air attract one another, although we have very little hope that experimental methods will ever be invented so delicate as to measure or even to make manifest this attraction. But we know that there is attraction between any portion of air and the earth, and we find by Cavendish's experiment that gravitating bodies, if of sufficient mass, gravitate sensibly towards each other, and we conclude that two portions of air gravitate towards each other. But it is still extremely doubtful whether the medium of light and electricity is a gravitating substance, though it is certainly material and has mass.

Cause of Gravitation.—Newton, in his *Principia*, deduces from the observed motions of the heavenly bodies the fact that they attract one another according to a definite law.

This he gives as a result of strict dynamical reasoning, and by it he shows how not only the more conspicuous phenomena, but all the apparent irregularities of the motions of these bodies are the calculable results of this single principle. In his *Principia* he confines himself to the demonstration and development of this great step in the science of the mutual action of bodies. He says nothing about the means by which bodies are made to gravitate towards each other. We know that his mind did not rest at this point—that he felt that gravitation itself must be capable of being explained, and that he even suggested an explanation depending on the action of an etherial medium pervading space. But with that wise moderation which is characteristic of all his investigations, he distinguished such speculations from what he had established by observation and demonstration, and excluded from his *Principia* all mention of the cause of gravitation, reserving his thoughts on this subject for the

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The attempts which have been made since the time of Newton to solve this difficult question are few in number, and have not led to any well-established result.

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If, on the other hand, we frame the hypothesis that the configuration, motion, or action of the material system is of a certain definite kind, and if the results of this hypothesis agree with the phenomena, then, unless we can prove

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